Exercise 7

(*difficult*) Solve $u_{tt} - c^2\Delta u = f(\mathbf{x})$, where $f(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, $f(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, A is a constant, and the initial data are identically zero. Sketch the regions in space-time that illustrate your answer. (*Hint:* Use (13) and find the volume of intersection of two balls, or use (11) and Exercise 9.2.6(b).)

Solution

We will start by solving the three-dimensional wave equation with a general inhomogeneous term and zero initial conditions.

$$\begin{aligned}
&u_{tt} - c^2\Delta u = f(x,y,z,t), \quad -\infty < x,y,z < \infty,\ -\infty < t < \infty \\
&u(x,y,z,0) = 0 \\
&u_t(x,y,z,0) = 0
\end{aligned}$$

According to Duhamel's principle, the solution to the inhomogeneous wave equation is

$$u(x,y,z,t) = \int_0^t U(x,y,z,t-s;s)\,ds,$$

where $U = U(x,y,z,t;s)$ is the solution to the associated homogeneous equation with a particular choice for the initial conditions.

$$\begin{aligned}
&U_{tt} - c^2\Delta U = 0, \quad -\infty < x,y,z < \infty,\ -\infty < t < \infty \\
&U(x,y,z,0;s) = 0 \\
&U_t(x,y,z,0;s) = f(x,y,z,s)
\end{aligned}$$

We can check that this formula for u satisfies the initial value problem. Differentiate u twice with respect to t by using the Leibnitz rule.

$$\begin{aligned}
u_t &= \int_0^t \frac{\partial}{\partial t}U(x,y,z,t-s;s)\,ds + U(x,y,z,0;t)\cdot 1 - U(x,y,z,t;0)\cdot 0 \\
&= \int_0^t U_t(x,y,z,t-s;s)\,ds + \underbrace{U(x,y,z,0;t)}_{=\,0} \\
&= \int_0^t U_t(x,y,z,t-s;s)\,ds \\
u_{tt} &= \int_0^t \frac{\partial}{\partial t}U_t(x,y,z,t-s;s)\,ds + U_t(x,y,z,0;t)\cdot 1 - U_t(x,y,z,t;0)\cdot 0 \\
&= \int_0^t U_{tt}(x,y,z,t-s;s)\,ds + \underbrace{U_t(x,y,z,0;t)}_{=\,f(x,y,z,t)} \\
&= \int_0^t c^2\Delta U(x,y,z,t-s;s)\,ds + f(x,y,z,t) \\
&= c^2\Delta\int_0^t U(x,y,z,t-s;s)\,ds + f(x,y,z,t) \\
&= c^2\Delta u + f(x,y,z,t)
\end{aligned}$$

As discussed in Section 9.2, the solution for U is given by the formula of Kirchhoff and Poisson.

$$U(x,y,z,t;s)=\frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} f(x_0,y_0,z_0,s)\,dS_0$$

Now u can be determined.

$$\begin{aligned}
u(x,y,z,t)&=\int_0^t U(x,y,z,t-s;s)\,ds\\
&=\int_0^t\left[\frac{1}{4\pi c^2(t-s)}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2(t-s)^2}} f(x_0,y_0,z_0,s)\,dS_0\right]ds\\
&=\frac{1}{4\pi c^2}\int_0^t\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2(t-s)^2}}\frac{f(x_0,y_0,z_0,s)}{t-s}\,dS_0\,ds
\end{aligned}$$

Write the surface integral over the sphere centered at (x,y,z) with radius $c(t-s)$ explicitly by using spherical coordinates (r_0,θ_0,ϕ_0), where ϕ_0 represents the angle from the polar axis.

$$\begin{aligned}
x_0-x&=c(t-s)\sin\theta_0\cos\phi_0\\
y_0-y&=c(t-s)\sin\theta_0\sin\phi_0\\
z_0-z&=c(t-s)\cos\theta_0
\end{aligned}$$

The solution becomes

$$\begin{aligned}
u(x,y,z,t)&=\frac{1}{4\pi c^2}\int_0^t\int_0^\pi\int_0^{2\pi}\frac{f(c(t-s),\theta_0,\phi_0,s)}{t-s}\,[c^2(t-s)^2\sin\phi_0\,d\theta_0\,d\phi_0]\,ds\\
&=\frac{1}{4\pi c^2}\int_0^t\int_0^\pi\int_0^{2\pi} f(c(t-s),\theta_0,\phi_0,s)\,[c^2(t-s)\sin\phi_0\,d\theta_0\,d\phi_0]\,ds.
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
r_0&=c(t-s)\quad\to\quad s=t-\frac{r_0}{c}\\
dr_0&=-c\,ds
\end{aligned}$$

As a result,

$$\begin{aligned}
u(x,y,z,t)&=\frac{1}{4\pi c^2}\int_{ct}^0\int_0^\pi\int_0^{2\pi} f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)(r_0\sin\phi_0\,d\theta_0\,d\phi_0)\,(-dr_0)\\
&=\frac{1}{4\pi c^2}\int_0^{ct}\int_0^\pi\int_0^{2\pi} f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)r_0\sin\phi_0\,d\theta_0\,d\phi_0\,dr_0\\
&=\frac{1}{4\pi c^2}\int_0^\pi\int_0^{2\pi}\int_0^{ct} f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0\\
&=\frac{1}{4\pi c^2}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\frac{f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)}{r_0}\,r_0^2\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0.
\end{aligned}$$

Note that

$$(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = r_0^2 \quad \to \quad r_0 = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}.$$

Changing back to Cartesian coordinates, the general solution for u is

$$\boxed{u(x, y, z, t) = \frac{1}{4\pi c^2} \iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2 \le c^2t^2}} \frac{f\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right)}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}}\, dV_0,}$$

which is essentially equation (13) in the text. In this exercise

$$f(x, y, z, t) = \begin{cases} A & \text{for } |\mathbf{x}| < \rho \\ 0 & \text{for } |\mathbf{x}| > \rho \end{cases},$$

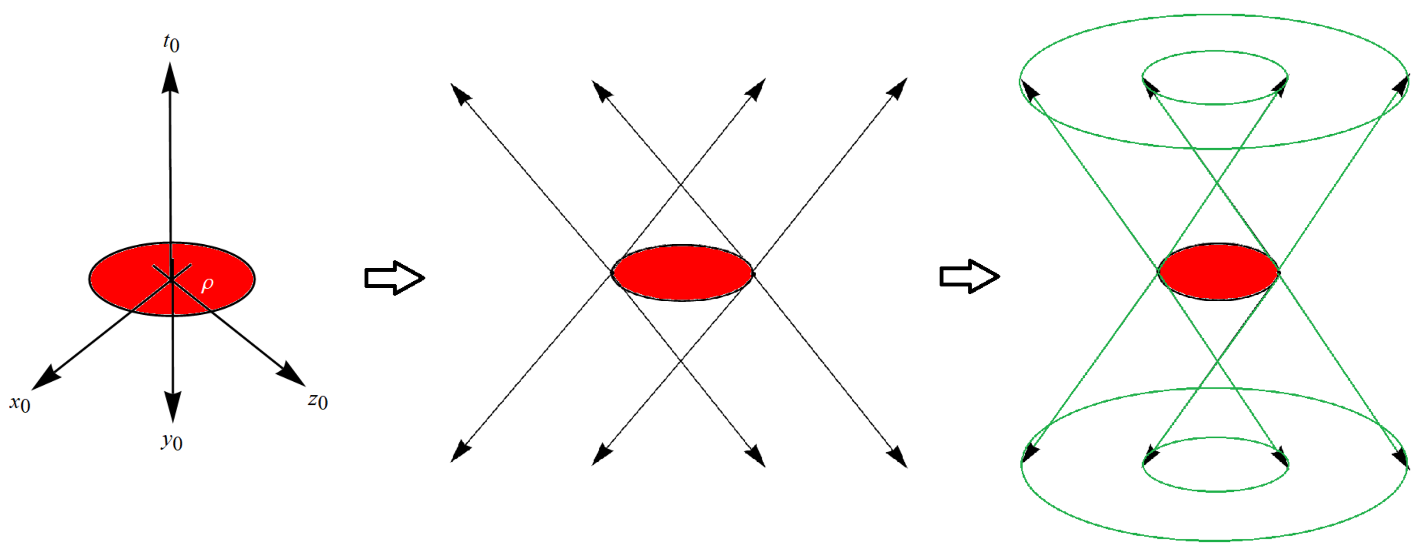
so the boxed formula simplifies to

$$u(x, y, z, t) = \frac{1}{4\pi c^2} \iiint\limits_{Q \cap T} \frac{A}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}}\, dV_0,$$

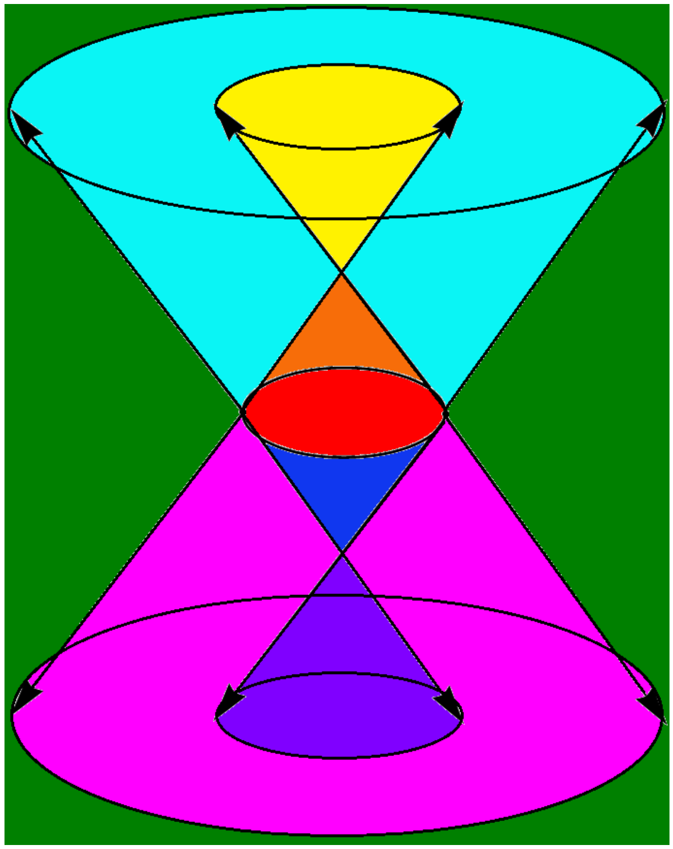
where

$$\begin{aligned} Q &= \{(x_0, y_0, z_0) \mid (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 \le c^2 t^2\} \\ T &= \{(x_0, y_0, z_0) \mid x_0^2 + y_0^2 + z_0^2 < \rho^2\}. \end{aligned}$$

Basically, this volume integral is over the part of the solid ball centered at (x, y, z) with radius $c|t|$ that lies within the solid ball centered at the origin with radius ρ. Depending what region in space-time the point (x, y, z, t) is chosen, the volume integral will yield a different result. The characteristic surfaces, which are obtained by drawing light rays (with slope c) from every point on the boundary of the red hyperdisk within which the inhomogeneous term is nonzero, separate these regions.

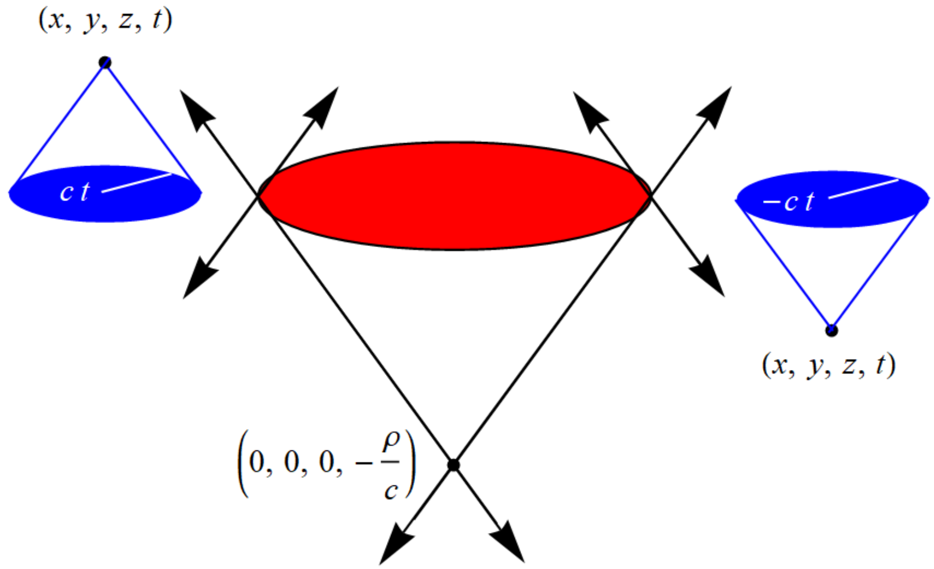


Let each of the regions in space-time be labelled by color.



Our aim now is to calculate the volume integral for u in each of the seven regions.

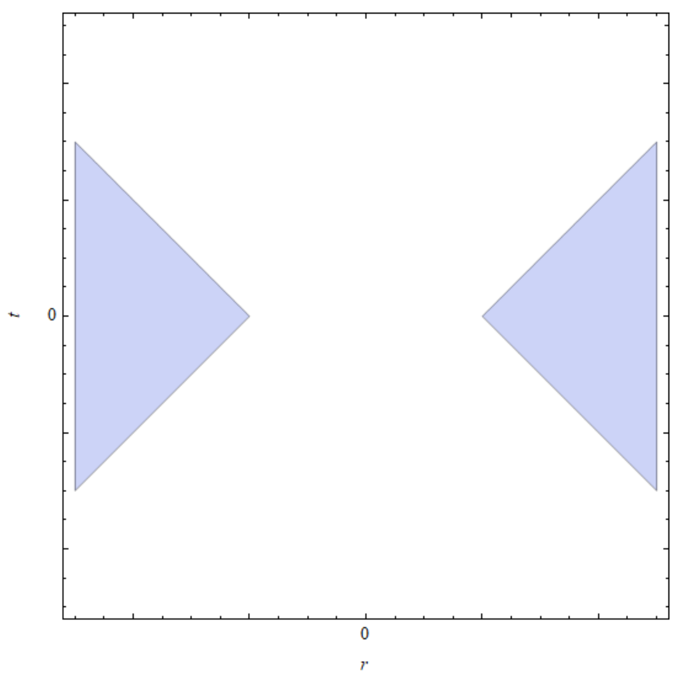
The Green Region



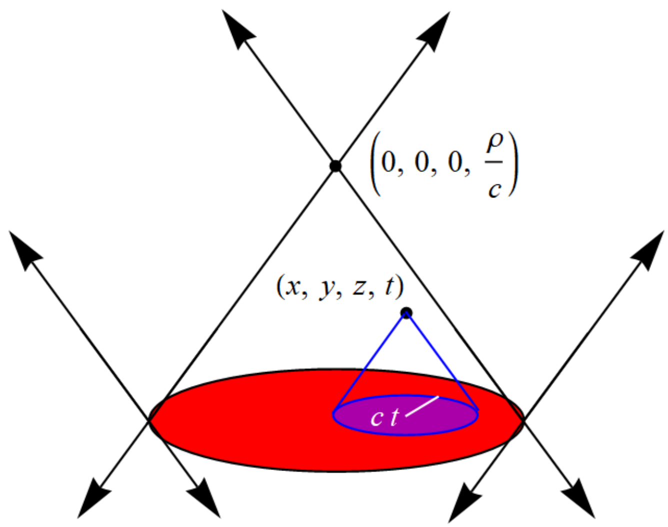
For points in the green region, regardless of whether $t > 0$ or $t < 0$, the blue balls do not intersect the red ball. As a result, $Q \cap T$ is the empty set, and the solution is zero.

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{4\pi c^2} \iiint\limits_{Q\cap T} \frac{A}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\, dV_0 \\ &= 0 \end{aligned}$$

This formula is valid for $r > \rho + c|t|$, where $r = \sqrt{x^2+y^2+z^2}$. The condition of validity in the rt-plane is roughly sketched below.



The Orange Region



The purple hyperdisk above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a solid ball centered at (x, y, z) with radius ct. The solution for u becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{4\pi c^2} \iiint_{Q\cap T} \frac{A}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\, dV_0 \\ &= \frac{A}{4\pi c^2} \iiint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2 \le c^2t^2}} \frac{dV_0}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}. \end{aligned}$$

Use spherical coordinates (r_0, θ_0, ϕ_0) to evaluate the integral.

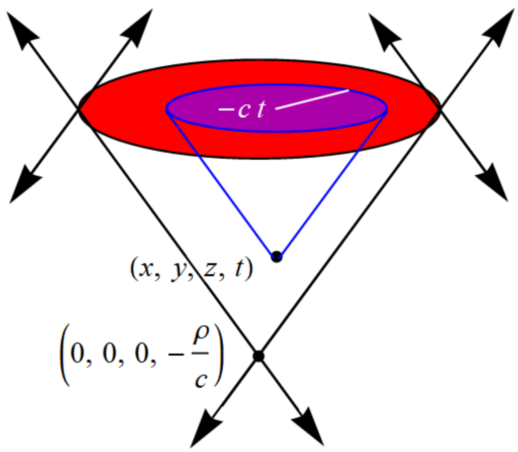
$$\begin{aligned} x_0 - x &= r_0 \sin\theta_0 \cos\phi_0 \\ y_0 - y &= r_0 \sin\theta_0 \sin\phi_0 \\ z_0 - z &= r_0 \cos\theta_0 \end{aligned}$$

Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2} \int_0^{\pi} \int_0^{2\pi} \int_0^{ct} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{r_0} \\ &= \frac{A}{4\pi c^2} \left(\int_0^{ct} r_0\, dr_0 \right) \left(\int_0^{2\pi} d\theta_0 \right) \left(\int_0^{\pi} \sin\phi_0\, d\phi_0 \right) \\ &= \frac{A}{4\pi c^2} \left(\frac{c^2t^2}{2} \right) (2\pi)(2) \\ &= \frac{1}{2} A t^2 \end{aligned}$$

in the orange region. This formula is valid for $r + ct < \rho$, where $r = \sqrt{x^2+y^2+z^2}$.

The Blue Region



The purple hyperdisk above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a solid ball centered at (x, y, z) with radius $-ct$. The solution for u becomes

$$\begin{aligned}
u(x,y,z,t) &= \frac{1}{4\pi c^2}\iiint\limits_{Q\cap T} \frac{A}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\,dV_0 \\
&= \frac{A}{4\pi c^2}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ +(z_0-z)^2\leq c^2t^2}} \frac{dV_0}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}.
\end{aligned}$$

Use spherical coordinates (r_0, θ_0, ϕ_0) to evaluate the integral.

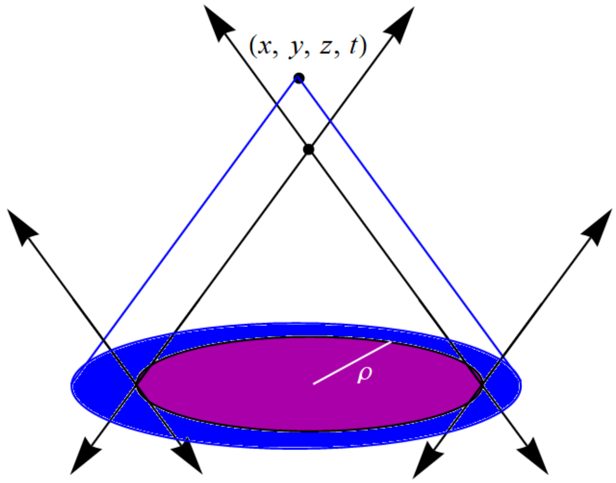
$$\begin{aligned}
x_0 - x &= r_0 \sin\theta_0 \cos\phi_0 \\
y_0 - y &= r_0 \sin\theta_0 \sin\phi_0 \\
z_0 - z &= r_0 \cos\theta_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{4\pi c^2}\int_0^{\pi}\int_0^{2\pi}\int_0^{-ct} \frac{r_0^2 \sin\phi_0\,dr_0\,d\theta_0\,d\phi_0}{r_0} \\
&= \frac{A}{4\pi c^2}\left(\int_0^{-ct} r_0\,dr_0\right)\left(\int_0^{2\pi} d\theta_0\right)\left(\int_0^{\pi}\sin\phi_0\,d\phi_0\right) \\
&= \frac{A}{4\pi c^2}\left(\frac{c^2t^2}{2}\right)(2\pi)(2) \\
&= \frac{1}{2}At^2
\end{aligned}$$

in the blue region. This formula is valid for $r - ct < \rho$, where $r = \sqrt{x^2+y^2+z^2}$.

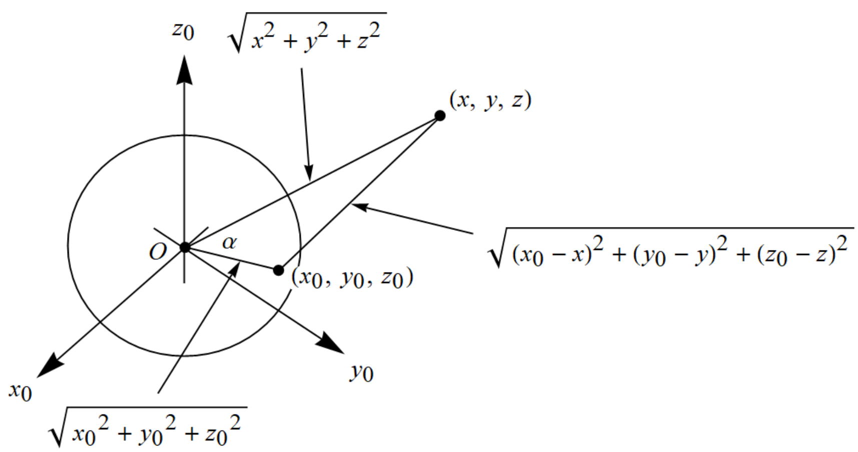
The Yellow Region



The purple hyperdisk above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a solid ball centered at the origin with radius ρ. The solution for u becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{4\pi c^2}\iiint_{Q\cap T}\frac{A}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\,dV_0 \\ &= \frac{A}{4\pi c^2}\iiint_{x_0^2+y_0^2+z_0^2\le\rho^2}\frac{dV_0}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}. \end{aligned}$$

Below is a visualization of the ball in $x_0y_0z_0$-space.



Notice that the denominator of the integrand is the distance from (x_0, y_0, z_0) to the point (x, y, z). Apply the law of cosines to the triangle above.

$$(x_0-x)^2+(y_0-y)^2+(z_0-z)^2 = (x_0^2+y_0^2+z_0^2)+(x^2+y^2+z^2)-2\sqrt{x^2+y^2+z^2}\sqrt{x_0^2+y_0^2+z_0^2}\cos\alpha$$

Take the square root of both sides.

$$\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2} = \sqrt{x_0^2+y_0^2+z_0^2-2\sqrt{x^2+y^2+z^2}\sqrt{x_0^2+y_0^2+z_0^2}\cos\alpha+x^2+y^2+z^2}$$

Substitute this result into the integrand, using $r = \sqrt{x^2 + y^2 + z^2}$ to make it more compact.

$$u(x, y, z, t) = \frac{A}{4\pi c^2} \iiint\limits_{x_0^2+y_0^2+z_0^2 \le \rho^2} \frac{dV_0}{\sqrt{x_0^2 + y_0^2 + z_0^2 - 2r\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha + r^2}}$$

Write the integral explicitly using spherical coordinates (r_0, θ_0, ϕ_0), where ϕ_0 is the angle from the polar axis.

$$\begin{aligned} x_0 &= r_0 \sin\theta_0 \cos\phi_0 \\ y_0 &= r_0 \sin\theta_0 \sin\phi_0 \\ z_0 &= r_0 \cos\theta_0 \end{aligned}$$

Since $x_0^2 + y_0^2 + z_0^2 = r_0^2$, we get

$$u(x, y, z, t) = \frac{A}{4\pi c^2} \int_0^\pi \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}.$$

Orient the polar axis in the direction of (x, y, z) to make $\alpha = \phi_0$. Doing so allows us to evaluate the integral in $d\phi_0$ with a substitution.

$$\begin{aligned} u(x, y, z, t) &= \frac{A}{4\pi c^2} \int_0^\pi \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \\ &= \frac{A}{4\pi c^2} \left(\int_0^{2\pi} d\theta_0 \right) \int_0^\pi \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \\ &= \frac{A}{2c^2} \int_0^\rho r_0 \left(\int_0^\pi \frac{r_0 \sin\phi_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \right) dr_0 \end{aligned}$$

Let

$$\begin{aligned} w &= r_0^2 - 2rr_0\cos\phi_0 + r^2 \\ dw &= 2rr_0 \sin\phi_0 \, d\phi_0 \quad \rightarrow \quad \frac{dw}{2r} = r_0 \sin\phi_0 \, d\phi_0. \end{aligned}$$

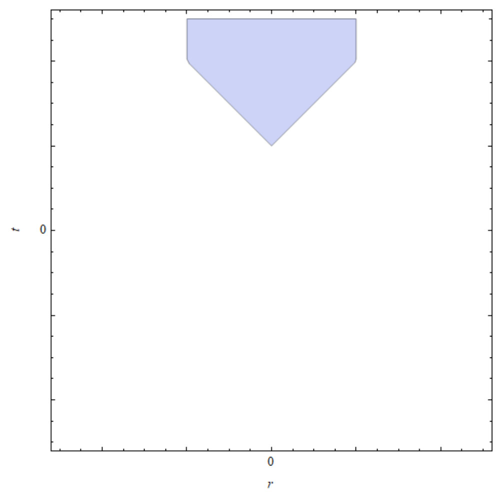
As a result,

$$\begin{aligned} u(x, y, z, t) &= \frac{A}{2c^2} \int_0^\rho r_0 \left[\int_{r_0^2 - 2rr_0 + r^2}^{r_0^2 + 2rr_0 + r^2} \frac{1}{\sqrt{w}} \left(\frac{dw}{2r} \right) \right] dr_0 \\ &= \frac{A}{2c^2 r} \int_0^\rho r_0 \left[\int_{(r_0 - r)^2}^{(r_0 + r)^2} \frac{dw}{2\sqrt{w}} \right] dr_0 \\ &= \frac{A}{2c^2 r} \int_0^\rho r_0 \left[\sqrt{(r_0 + r)^2} - \sqrt{(r_0 - r)^2} \right] dr_0 \\ &= \frac{A}{2c^2 r} \int_0^\rho r_0 \left(r_0 + r - |r_0 - r| \right) dr_0. \end{aligned}$$

Depending whether the point (x, y, z) is inside or outside the ball $x_0^2 + y_0^2 + z_0^2 \le \rho^2$, the result for u will be different because of the absolute value sign. The yellow region is thus divided into two subregions, the set of points inside the cylinder $r_0 = \rho$ and the set of points outside of it. If (x, y, z) is inside, that is, $r = \sqrt{x^2 + y^2 + z^2} < \rho$, then

$$\begin{aligned} u(x, y, z, t) &= \frac{A}{2c^2 r}\left\{\int_0^r r_0\left[r_0 + r - (r - r_0)\right] dr_0 + \int_r^\rho r_0\left[r_0 + r - (r_0 - r)\right] dr_0\right\} \\ &= \frac{A}{2c^2 r}\left[\frac{1}{3}r(3\rho^2 - r^2)\right] = \frac{A}{6c^2}(3\rho^2 - r^2). \end{aligned}$$

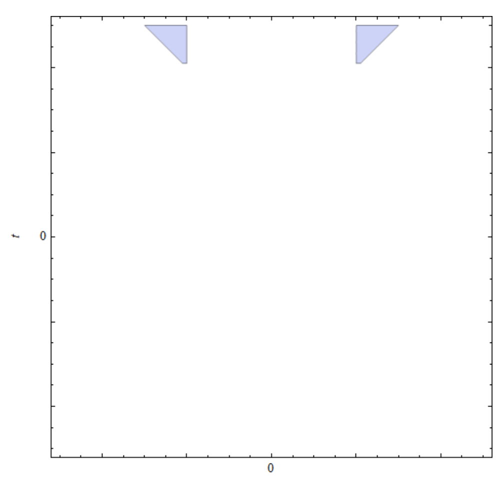
This result is valid if $ct > r + \rho$ and $r < \rho$. A rough sketch of this validity condition in the rt-plane is shown below.



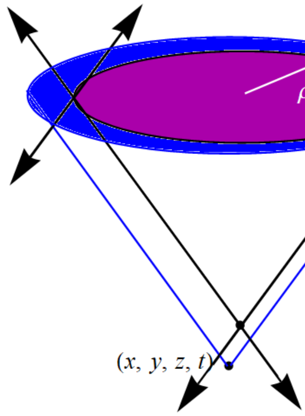
On the other hand, if (x, y, z) is outside the ball, that is, $r = \sqrt{x^2 + y^2 + z^2} > \rho$, then

$$\begin{aligned} u(x, y, z, t) &= \frac{A}{2c^2 r}\int_0^\rho r_0\left[r_0 + r - (r - r_0)\right] dr_0 \\ &= \frac{A}{2c^2 r}\left(\frac{2\rho^3}{3}\right) = \frac{A\rho^3}{3c^2 r}. \end{aligned}$$

This result is valid if $ct > r + \rho$ and $r > \rho$. A rough sketch of this validity condition in the rt-plane is shown below.



The Purple Region



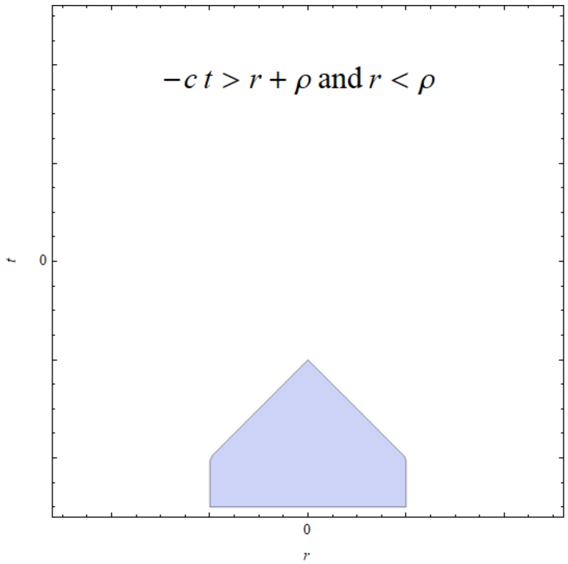
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a solid ball centered at the origin with radius ρ. The solution for u becomes

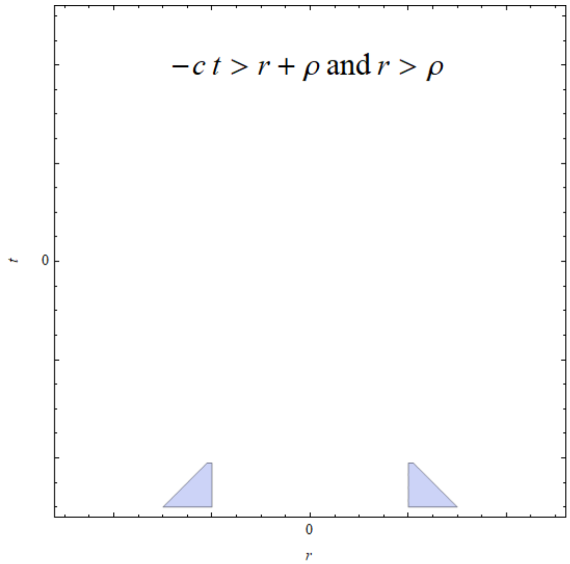
$$\begin{aligned} u(x,y,z,t) &= \frac{1}{4\pi c^2} \iiint\limits_{Q\cap T} \frac{A}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\, dV_0 \\ &= \frac{A}{4\pi c^2} \iiint\limits_{x_0^2+y_0^2+z_0^2\le\rho^2} \frac{dV_0}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}, \end{aligned}$$

which is the same as the one for the yellow region.

$$u(x,y,z,t) = \begin{cases} \dfrac{A}{6c^2}(3\rho^2 - r^2) & \text{if } -ct > r+\rho \text{ and } r < \rho \\ \dfrac{A\rho^3}{3c^2 r} & \text{if } -ct > r+\rho \text{ and } r > \rho \end{cases}$$

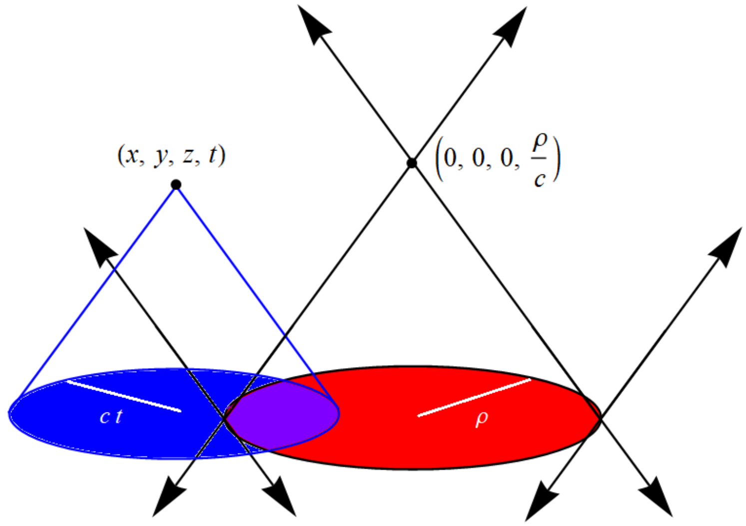
As before, the purple region is divided into two subregions. The conditions of validity in the rt-plane are roughly sketched below.



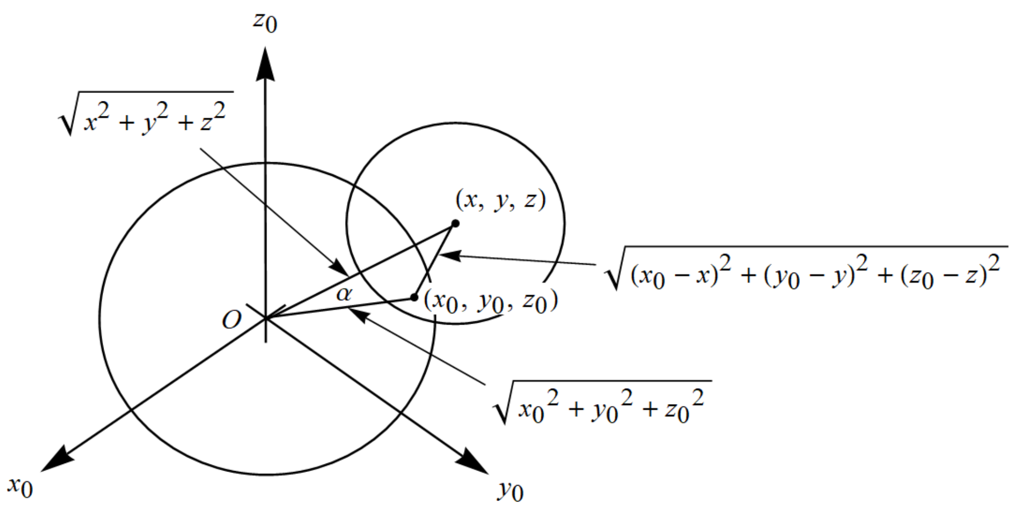


The Cyan Region - Position 1 of 4: $r > \rho$ and $r - ct > 0$

Several different intersections can occur in the cyan region. In this first part we will consider a point in space-time (x, y, z, t) outside the cylinder $r = \rho$ and under the cone $r - ct = 0$.



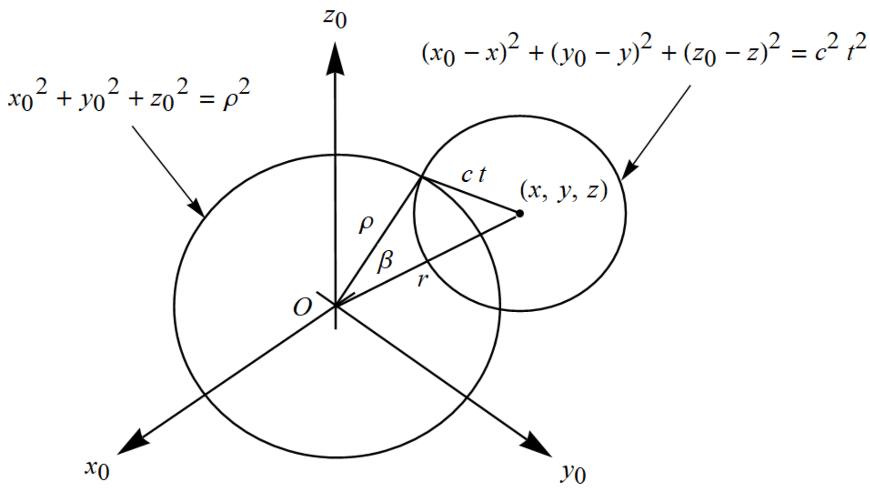
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves of the balls and the maximum polar angle β.



Use the law of cosines to find β.

$$c^2t^2 = r^2 + \rho^2 - 2\rho r\cos\beta \quad \to \quad \cos\beta = \frac{r^2 + \rho^2 - c^2t^2}{2\rho r}$$

The bounding curves are as follows.

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= \rho^2 \\ r_0^2 &= \rho^2 \\ r_0 &= \rho \end{aligned}$$

$$\begin{aligned} (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= c^2t^2 \\ r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0^2 - (2r\cos\alpha)r_0 + (r^2 - c^2t^2) &= 0 \\ r_0 = \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha - 4(r^2 - c^2t^2)}}{2} \\ r_0 = r\cos\alpha \pm \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)} \\ r_0 = r\cos\alpha - \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)} \end{aligned}$$

The minus sign was chosen because we want the part of the bounding curve closest to O. As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x, y, z, t) &= \frac{A}{4\pi c^2} \iiint\limits_{Q\cap T} \frac{dV_0}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}} \\ &= \frac{A}{4\pi c^2} \int_0^\beta \int_0^{2\pi} \int_{r\cos\alpha - \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}}^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}. \end{aligned}$$

Orienting the polar axis as we have makes it so that $\alpha = \phi_0$.

$$u(x, y, z, t) = \frac{A}{4\pi c^2} \int_0^\beta \int_0^{2\pi} \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}$$

Proceed to evaluate the integral in $d\theta_0$.

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2}\left(\int_0^{2\pi} d\theta_0\right)\int_0^\beta \int_{r\cos\phi_0-\sqrt{r^2\cos^2\phi_0-(r^2-c^2t^2)}}^\rho \frac{r_0^2\sin\phi_0\,dr_0\,d\phi_0}{\sqrt{r_0^2-2rr_0\cos\phi_0+r^2}} \\ &= \frac{A}{2c^2}\int_0^\beta \int_{r\cos\phi_0-\sqrt{r^2\cos^2\phi_0-(r^2-c^2t^2)}}^\rho \frac{r_0^2\sin\phi_0\,dr_0\,d\phi_0}{\sqrt{r_0^2-2rr_0\cos\phi_0+r^2}} \end{aligned}$$

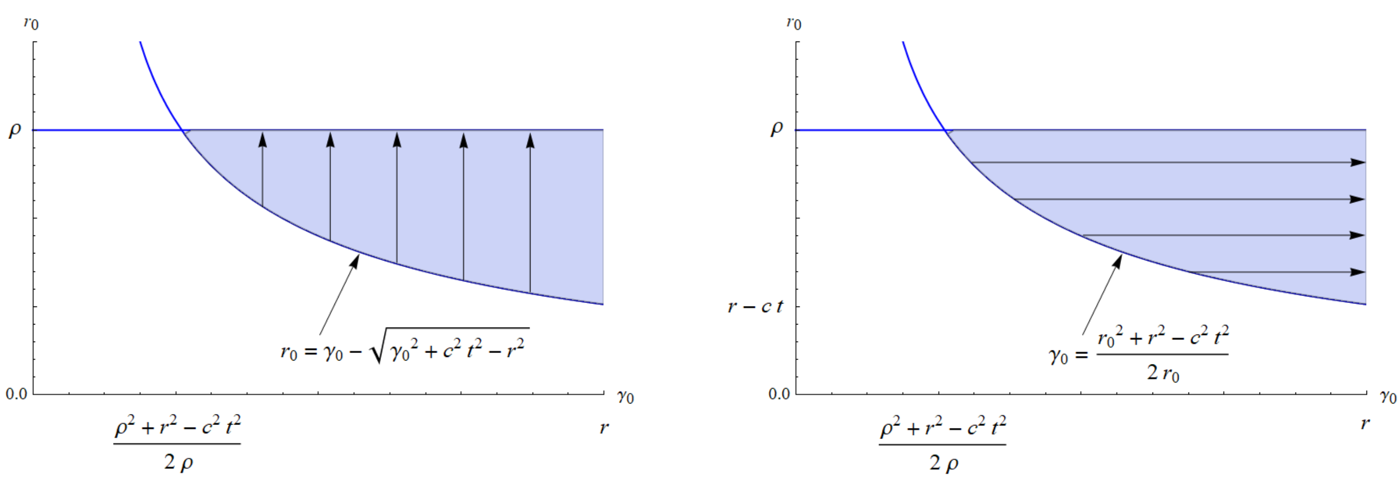
Make the following change of variables.

$$\begin{aligned} \gamma_0 &= r\cos\phi_0 \\ d\gamma_0 &= -r\sin\phi_0\,d\phi_0 \quad \to \quad -\frac{d\gamma_0}{r} = \sin\phi_0\,d\phi_0 \end{aligned}$$

Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{2c^2}\int_r^{r\cos\beta}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^\rho \frac{r_0^2\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\left(-\frac{d\gamma_0}{r}\right) \\ &= \frac{A}{2c^2r}\int_{r\cos\beta}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^\rho \frac{r_0^2\,dr_0\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} \\ &= \frac{A}{2c^2r}\int_{\frac{\rho^2+r^2-c^2t^2}{2\rho}}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^\rho \frac{r_0^2\,dr_0\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}. \end{aligned} \tag{1}$$

Suppose first that $r^2 > c^2t^2+\rho^2$. Then the current mode of integration in the $\gamma_0 r_0$-plane is shown on the left.



Integrate over the domain as shown on the right in order to switch the order of integration.

$$u(x,y,z,t) = \frac{A}{2c^2r}\int_{r-ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}$$

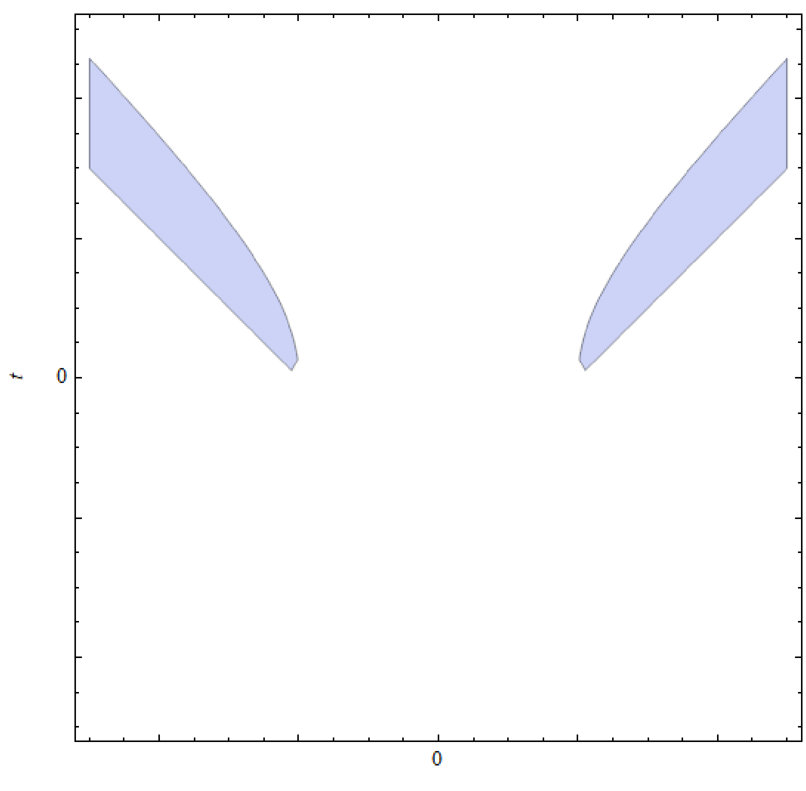
Make the following substitution.

$$\begin{aligned} s_0 &= r_0^2-2r_0\gamma_0+r^2 \\ ds_0 &= -2r_0\,d\gamma_0 \quad \to \quad -\frac{ds_0}{2} = r_0\,d\gamma_0 \end{aligned}$$

Therefore,

$$
\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2r}\int_{r-ct}^{\rho} r_0 \left(\int_{c^2t^2}^{r_0^2-2rr_0+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right) dr_0 \\
&= \frac{A}{2c^2r}\int_{r-ct}^{\rho} r_0 \left[\int_{(r_0-r)^2}^{c^2t^2} \frac{ds_0}{2\sqrt{s_0}}\right] dr_0 \\
&= \frac{A}{2c^2r}\int_{r-ct}^{\rho} r_0 \left[\sqrt{c^2t^2} - \sqrt{(r_0-r)^2}\right] dr_0 \\
&= \frac{A}{2c^2r}\int_{r-ct}^{\rho} r_0 \left(ct - |r_0 - r|\right) dr_0 \\
&= \frac{A}{2c^2r}\int_{r-ct}^{\rho} r_0 [ct - (r - r_0)]\, dr_0 \\
&= \frac{A}{2c^2r}\left\{\frac{1}{6}\left[r^3 - c^3t^3 + 2\rho^3 - 3ct(r^2-\rho^2) + 3r(c^2t^2-\rho^2)\right]\right\} \\
&= \frac{A}{12c^2r}\left[r^3 - c^3t^3 + 2\rho^3 - 3ct(r^2-\rho^2) + 3r(c^2t^2-\rho^2)\right].
\end{aligned}
$$

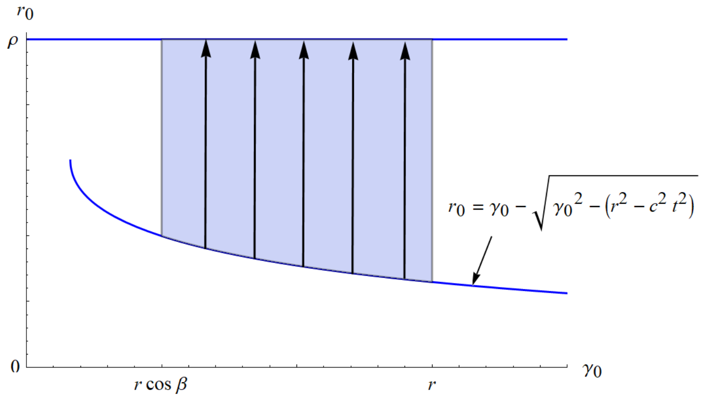
This solution is valid for $|\rho - ct| < r < \rho + ct$ and $r > \rho$ and $r - ct > 0$ and $r^2 > c^2t^2 + \rho^2$. A rough sketch of the validity condition in the rt-plane is shown below.

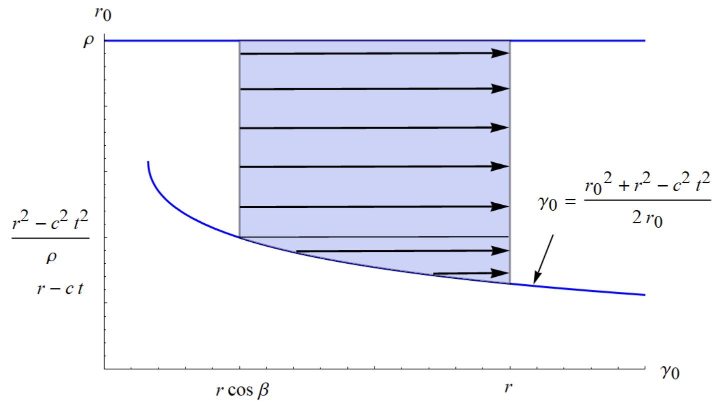


Return to equation (1).

$$u(x,y,z,t) = \frac{A}{2c^2r}\int_{\frac{\rho^2+r^2-c^2t^2}{2\rho}}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\rho}\frac{r_0^2\,dr_0\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} \tag{1}$$

Suppose secondly that $r^2 < c^2t^2+\rho^2$. Then the current mode of integration in the $\gamma_0 r_0$-plane is shown on the left.





Integrate over the domain as shown on the right in order to switch the order of integration.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_{r-ct}^{\frac{r^2-c^2t^2}{\rho}}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r}\frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho}\int_{r\cos\beta}^{r}\frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\right]$$

Make the following substitution.

$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$
$$ds_0 = -2r_0\,d\gamma_0 \quad\to\quad -\frac{ds_0}{2} = r_0\,d\gamma_0$$

As a result,

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2r}\left[\int_{r-ct}^{\frac{r^2-c^2t^2}{\rho}} r_0\left(\int_{c^2t^2}^{r_0^2-2rr_0+r^2}\frac{-ds_0}{2\sqrt{s_0}}\right)dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\left(\int_{r_0^2-2r_0r\cos\beta+r^2}^{r_0^2-2rr_0+r^2}\frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\right]\\
&= \frac{A}{2c^2r}\left[\int_{r-ct}^{\frac{r^2-c^2t^2}{\rho}} r_0\,(ct-|r_0-r|)\,dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\left(\sqrt{r_0^2-2r_0r\cos\beta+r^2}-|r_0-r|\right)dr_0\right]\\
&= \frac{A}{2c^2r}\left[\int_{r-ct}^{\frac{r^2-c^2t^2}{\rho}} r_0ct\,dr_0-\int_{r-ct}^{\rho} r_0|r_0-r|\,dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2-2r_0r\cos\beta+r^2}\,dr_0\right]\\
&= \frac{A}{2c^2r}\left[\int_{r-ct}^{\frac{r^2-c^2t^2}{\rho}} r_0ct\,dr_0-\int_{r-ct}^{\rho} r_0(r-r_0)\,dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2-2r_0r\cos\beta+r^2}\,dr_0\right]\\
&= \frac{A}{2c^2r}\left[\frac{3ct(r^2-c^2t^2)^2+(r-ct)^3\rho^2-3r\rho^4+2\rho^5}{6\rho^2}+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2-2r_0r\cos\beta+r^2}\,dr_0\right].
\end{aligned}$$

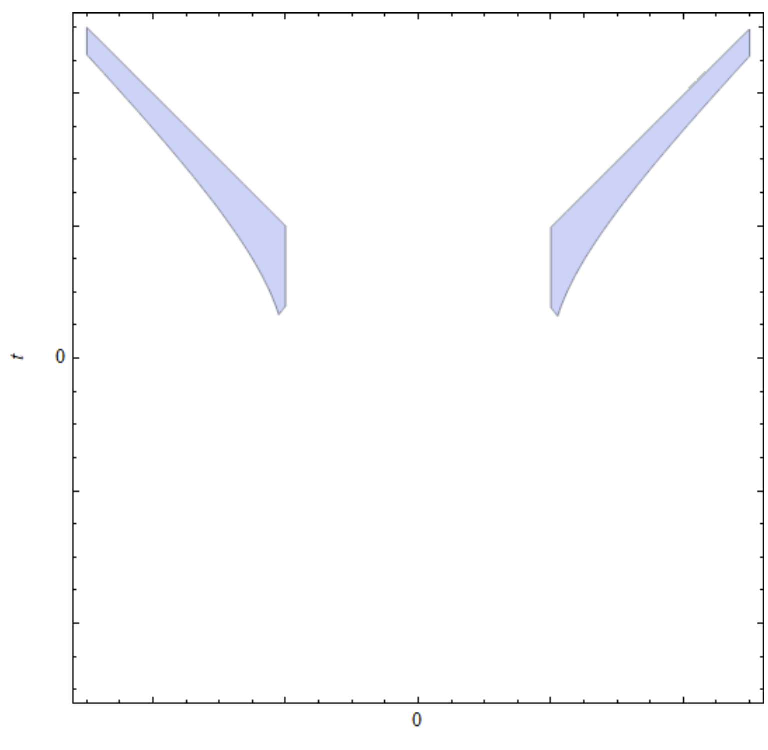
Proof for the result of this final integral will be given at the end.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left\{\frac{3ct(r^2-c^2t^2)^2+(r-ct)^3\rho^2-3r\rho^4+2\rho^5}{6\rho^2}\right.$$
$$+\frac{(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]}{8\rho^3}\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}$$
$$\left.+\frac{ct[\rho^4-(r^2-c^2t^2)^2]}{4\rho^2}\right\}$$
$$= \frac{A}{2c^2r}\left\{\frac{(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]}{8\rho^3}\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}\right.$$
$$\left.+\frac{3ct(r^2-c^2t^2)^2+2\rho^2(r-ct)^3-3\rho^4(2r-ct)+4\rho^5}{12\rho^2}\right\}$$

Therefore,

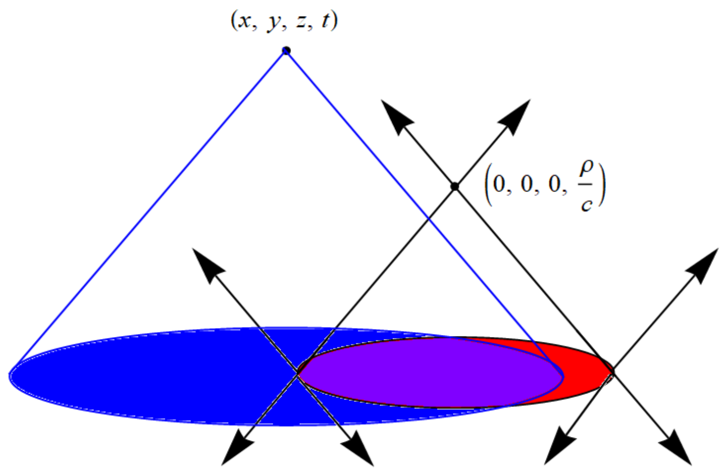
$$u(x,y,z,t) = \frac{A}{48\rho^3c^2r}\left\{3(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}\right.$$
$$\left.+2\rho\left[3ct(r^2-c^2t^2)^2+2\rho^2(r-ct)^3-3\rho^4(2r-ct)+4\rho^5\right]\right\}.$$

This solution is valid for $|\rho-ct|<r<\rho+ct$ and $r>\rho$ and $r-ct>0$ and $r^2<c^2t^2+\rho^2$. A rough sketch of the validity condition in the rt-plane is shown below.

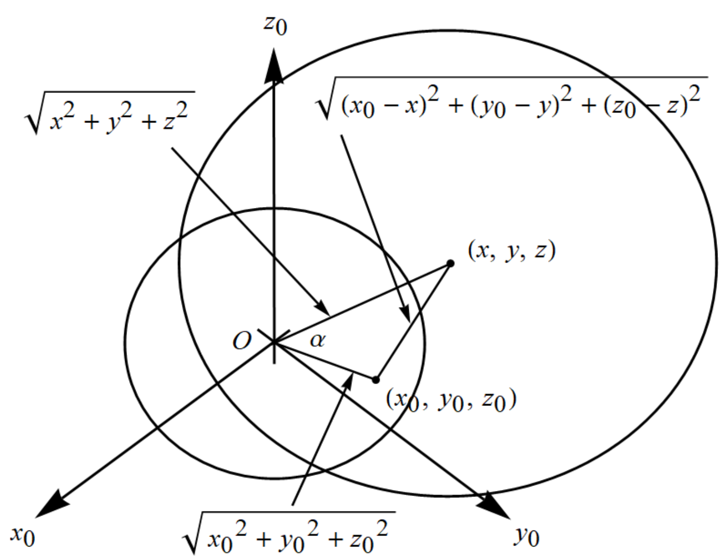


The Cyan Region - Position 2 of 4: $r > \rho$ and $r - ct < 0$

Several different intersections can occur in the cyan region. In this second part we will consider a point in space-time (x, y, z, t) outside the cylinder $r = \rho$ and above the cone $r - ct = 0$.



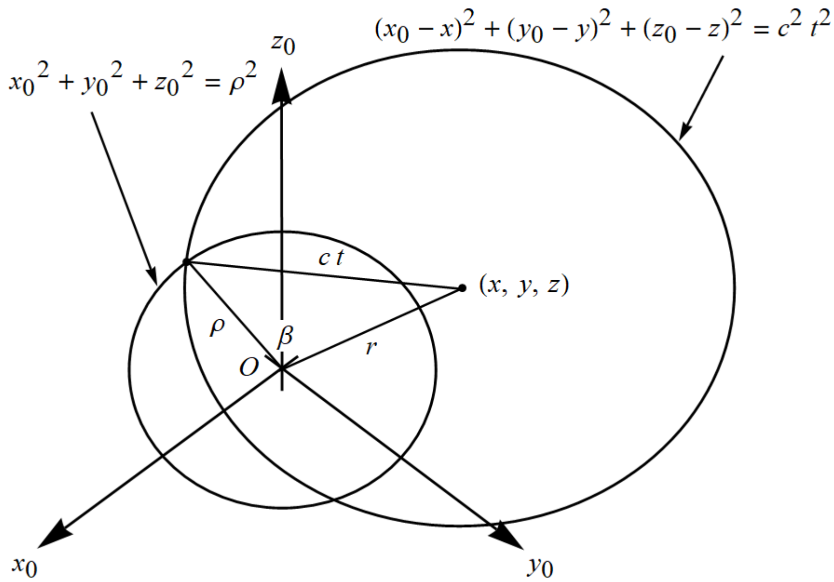
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves and the angle β at which one curve changes to the other.



Use the law of cosines again.

$$c^2t^2 = \rho^2 + r^2 - 2\rho r\cos\beta \quad \rightarrow \quad \cos\beta = \frac{\rho^2 + r^2 - c^2t^2}{2\rho r}$$

The bounding curves are

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= \rho^2 \\ r_0^2 &= \rho^2 \\ r_0 &= \rho \end{aligned} \qquad\qquad \begin{aligned} (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= c^2t^2 \\ r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0^2 - (2r\cos\alpha)r_0 - (c^2t^2 - r^2) &= 0 \\ r_0 &= \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha + 4(c^2t^2 - r^2)}}{2} \\ r_0 &= r\cos\alpha \pm \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)} \\ r_0 &= r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)}. \end{aligned}$$

The positive sign is chosen because r_0, the distance to (x_0, y_0, z_0), must be positive. As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x, y, z, t) &= \frac{A}{4\pi c^2} \iiint\limits_{Q\cap T} \frac{dV_0}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}} \\ &= \frac{A}{4\pi c^2}\left[\int_0^\beta \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} \right. \\ &\qquad \left. + \int_\beta^\pi \int_0^{2\pi} \int_0^{r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}\right]. \end{aligned}$$

Orienting the polar axis in the direction of (x, y, z) makes it so that $\alpha = \phi_0$.

$$\begin{aligned} u(x,y,z,t) = \frac{A}{4\pi c^2} \Bigg[& \int_0^\beta \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0 \cos\phi_0 + r^2}} \\ & + \int_\beta^\pi \int_0^{2\pi} \int_0^{r\cos\phi_0 + \sqrt{r^2 \cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0 \cos\phi_0 + r^2}} \Bigg] \end{aligned}$$

Evaluate the integral in $d\theta_0$.

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2} \Bigg[\left(\int_0^{2\pi} d\theta_0 \right) \int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0 \cos\phi_0 + r^2}} \\ &\qquad + \left(\int_0^{2\pi} d\theta_0 \right) \int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2 \cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0 \cos\phi_0 + r^2}} \Bigg] \\ &= \frac{A}{2c^2} \Bigg[\int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0 \cos\phi_0 + r^2}} \\ &\qquad + \int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2 \cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0 \cos\phi_0 + r^2}} \Bigg] \end{aligned}$$

Make the following substitution in both integrals.

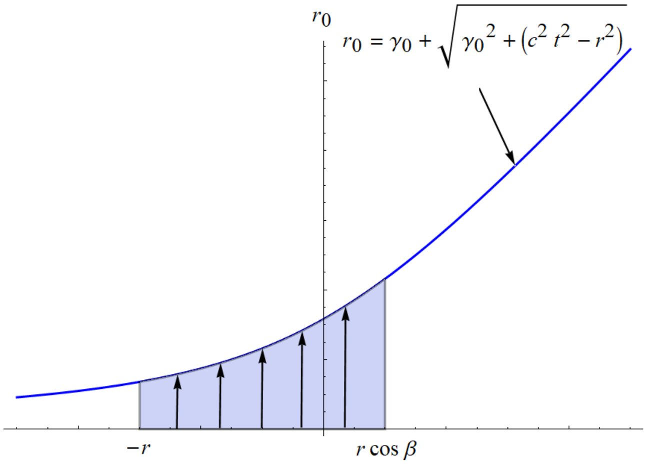
$$\begin{aligned} \gamma_0 &= r\cos\phi_0 \\ d\gamma_0 &= -r\sin\phi_0 \, d\phi_0 \quad \rightarrow \quad -\frac{d\gamma_0}{r} = \sin\phi_0 \, d\phi_0 \end{aligned}$$

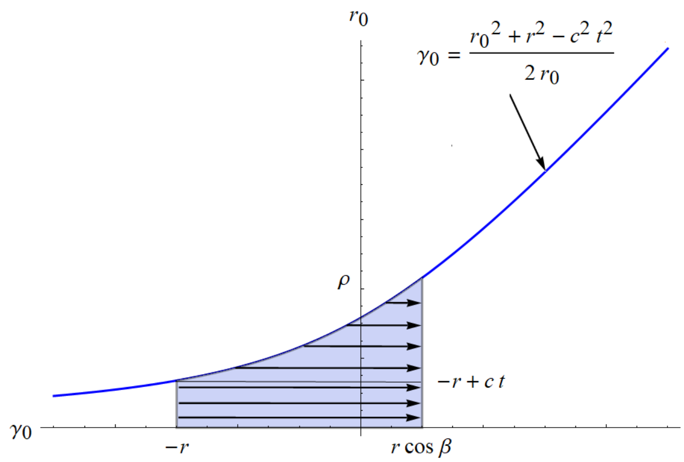
Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{2c^2} \Bigg[\int_r^{r\cos\beta} \int_0^\rho \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \left(-\frac{d\gamma_0}{r} \right) \\ &\qquad + \int_{r\cos\beta}^{-r} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \left(-\frac{d\gamma_0}{r} \right) \Bigg] \\ &= \frac{A}{2c^2 r} \Bigg[\int_{r\cos\beta}^r \int_0^\rho \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \\ &\qquad + \int_{-r}^{r\cos\beta} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \Bigg]. \end{aligned}$$

The order of integration in both integrals will now be switched.

Suppose that $r^2 < c^2t^2 + \rho^2$. Then the current mode of integration in the second integral is shown on the left.





Integrate over the domain as shown on the right to switch the order of the second integral.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_0^\rho \int_{r\cos\beta}^r \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} + \int_0^{-r+ct}\int_{-r}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \right.$$
$$\left. + \int_{-r+ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right]$$

Make the following substitution in all three integrals.

$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$
$$ds_0 = -2r_0\,d\gamma_0 \quad \to \quad -\frac{ds_0}{2} = r_0\,d\gamma_0$$

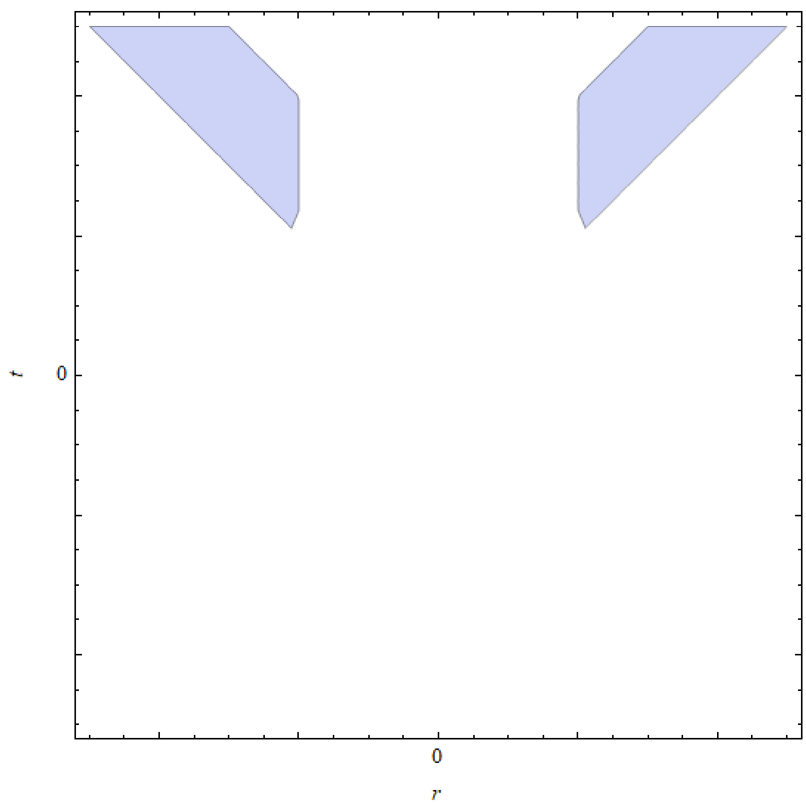
Consequently,

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\int_{r_0^2-2rr_0\cos\beta+r^2}^{r_0^2-2rr_0+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0 \right.$$
$$+ \int_0^{-r+ct} r_0\left(\int_{r_0^2+2rr_0+r^2}^{r_0^2-2rr_0\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0$$
$$\left. + \int_{-r+ct}^{\rho} r_0\left(\int_{c^2t^2}^{r_0^2-2rr_0\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\right]$$
$$= \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\sqrt{r_0^2 - 2rr_0\cos\beta + r^2} - |r_0 - r|\right)dr_0 \right.$$
$$+ \int_0^{-r+ct} r_0\left(r_0 + r - \sqrt{r_0^2 - 2rr_0\cos\beta + r^2}\right)dr_0$$
$$\left. + \int_{-r+ct}^{\rho} r_0\left(ct - \sqrt{r_0^2 - 2rr_0\cos\beta + r^2}\right)dr_0\right].$$

Therefore,

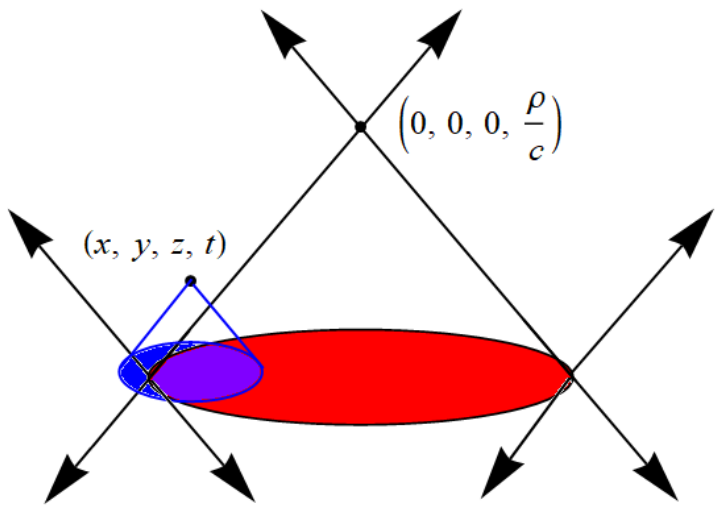
$$
\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2 r}\left[\int_0^\rho r_0\sqrt{r_0^2 - 2rr_0\cos\beta + r^2}\,dr_0 - \int_0^\rho r_0|r_0 - r|\,dr_0 + \int_0^{-r+ct} r_0(r_0+r)\,dr_0\right. \\
&\qquad - \int_0^{-r+ct} r_0\sqrt{r_0^2 - 2rr_0\cos\beta + r^2}\,dr_0 + \int_{-r+ct}^{\rho} r_0 ct\,dr_0 \\
&\qquad\qquad \left. - \int_{-r+ct}^{\rho} r_0\sqrt{r_0^2 - 2rr_0\cos\beta + r^2}\,dr_0\right] \\
&= \frac{A}{2c^2 r}\left[-\int_0^\rho r_0|r_0 - r|\,dr_0 + \int_0^{-r+ct} r_0(r_0+r)\,dr_0 + \int_{-r+ct}^{\rho} r_0 ct\,dr_0\right] \\
&= \frac{A}{2c^2 r}\left[-\int_0^\rho r_0(r - r_0)\,dr_0 + \int_0^{-r+ct} r_0(r_0+r)\,dr_0 + \int_{-r+ct}^{\rho} r_0 ct\,dr_0\right] \\
&= \frac{A}{2c^2 r}\left\{\frac{1}{6}\left[r^3 - c^3t^3 + 2\rho^3 - 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right]\right\} \\
&= \frac{A}{12c^2 r}\left[r^3 - c^3t^3 + 2\rho^3 - 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right].
\end{aligned}
$$

This solution is valid for $|\rho - ct| < r < \rho + ct$ and $r > \rho$ and $r - ct < 0$ and $r^2 < c^2t^2 + \rho^2$. There's no need to consider the alternative case of $r^2 > c^2t^2 + \rho^2$ because there are no points in space-time that satisfy $|\rho - ct| < r < \rho + ct$ and $r > \rho$ and $r - ct < 0$ and $r^2 > c^2t^2 + \rho^2$. Having said that, the preceding condition of validity can be relaxed to $|\rho - ct| < r < \rho + ct$ and $r > \rho$ and $r - ct < 0$. A rough sketch of it in the rt-plane is shown below.

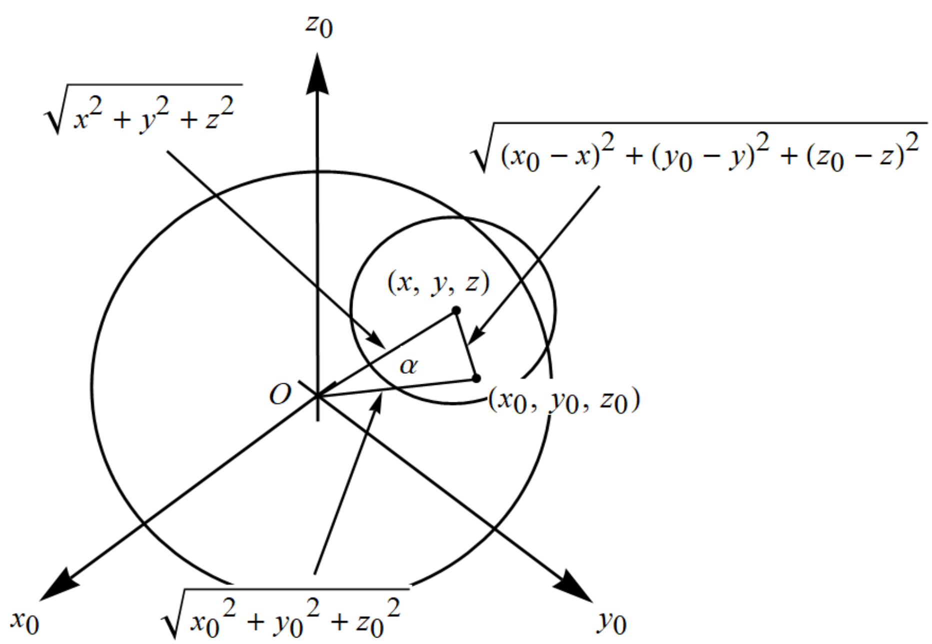


The Cyan Region - Position 3 of 4: $r < \rho$ and $r - ct > 0$

Several different intersections can occur in the cyan region. In this third part we will consider a point in space-time (x, y, z, t) inside the cylinder $r = \rho$ and below the cone $r - ct = 0$.



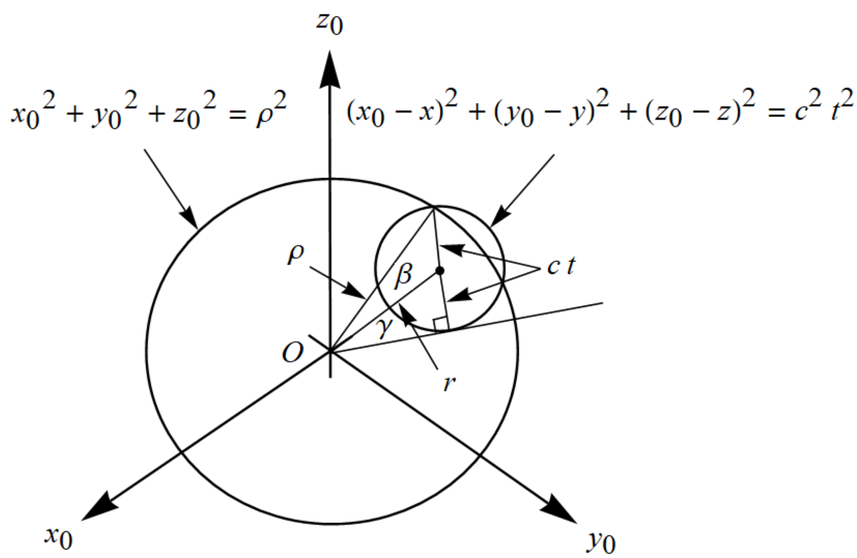
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves and the angles, β (the angle at which one curve changes to another) and γ (the maximum polar angle).



Use the law of cosines in the upper triangle to find β.

$$c^2t^2 = r^2 + \rho^2 - 2\rho r\cos\beta \quad \rightarrow \quad \cos\beta = \frac{r^2 + \rho^2 - c^2t^2}{2\rho r}$$

According to the Pythagorean theorem, the length of the missing side of the lower triangle is $r\cos\gamma = \sqrt{r^2 - c^2t^2}$. The bounding curves are

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= \rho^2 & (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= c^2t^2 \\ r_0^2 &= \rho^2 & r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0 &= \rho & r_0^2 - (2r\cos\alpha)r_0 + (r^2 - c^2t^2) &= 0 \\ && r_0 &= \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha - 4(r^2 - c^2t^2)}}{2} \\ && r_0 &= r\cos\alpha \pm \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}. \end{aligned}$$

As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2} \iiint_{Q\cap T} \frac{dV_0}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}} \\ &= \frac{A}{4\pi c^2} \left[\int_0^\gamma \int_0^{2\pi} \int_{r\cos\alpha - \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}}^{r\cos\alpha + \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} \right. \\ &\qquad \left. - \int_0^\beta \int_0^{2\pi} \int_\rho^{r\cos\alpha + \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} \right]. \end{aligned}$$

What we're essentially doing is integrating over the whole ball of radius ct and then subtracting the volume integral over the part that lies outside the ball of radius ρ.

Orienting the polar axis as we have makes it so that $\alpha = \phi_0$.

$$u(x,y,z,t) = \frac{A}{4\pi c^2}\left[\int_0^\gamma \int_0^{2\pi} \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \right.$$
$$\left. - \int_0^\beta \int_0^{2\pi} \int_\rho^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right]$$

Proceed to evaluate the integral in $d\theta_0$.

$$u(x,y,z,t) = \frac{A}{4\pi c^2}\left[\left(\int_0^{2\pi} d\theta_0\right) \int_0^\gamma \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \right.$$
$$\left. - \left(\int_0^{2\pi} d\theta_0\right) \int_0^\beta \int_\rho^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right]$$
$$= \frac{A}{2c^2}\left[\int_0^\gamma \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \right.$$
$$\left. - \int_0^\beta \int_\rho^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2-c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right]$$

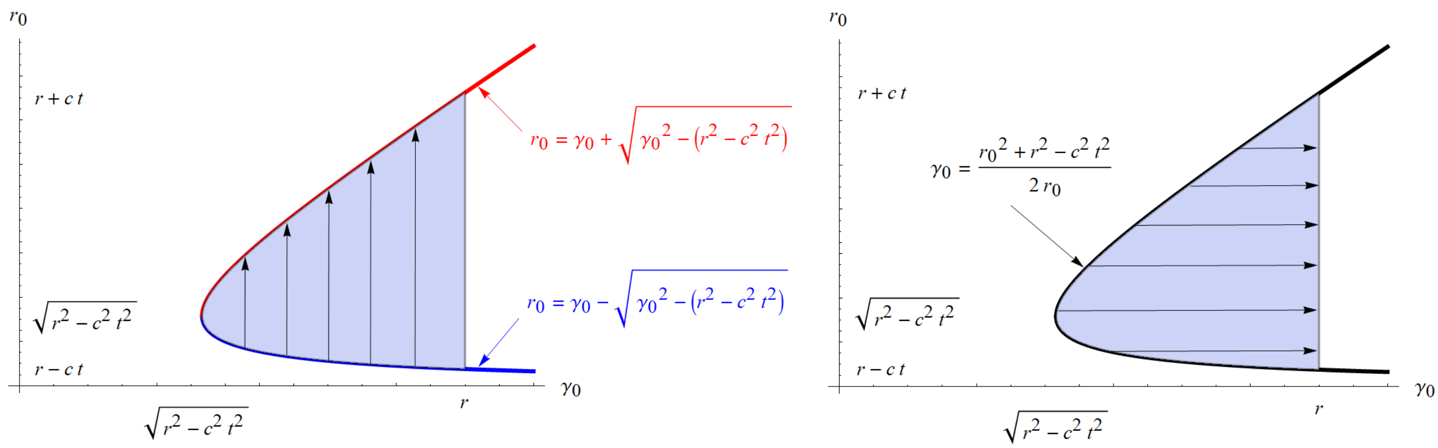
Make the following substitution in both integrals.

$$\gamma_0 = r\cos\phi_0$$
$$d\gamma_0 = -r\sin\phi_0 \, d\phi_0 \quad \rightarrow \quad -\frac{d\gamma_0}{r} = \sin\phi_0 \, d\phi_0$$

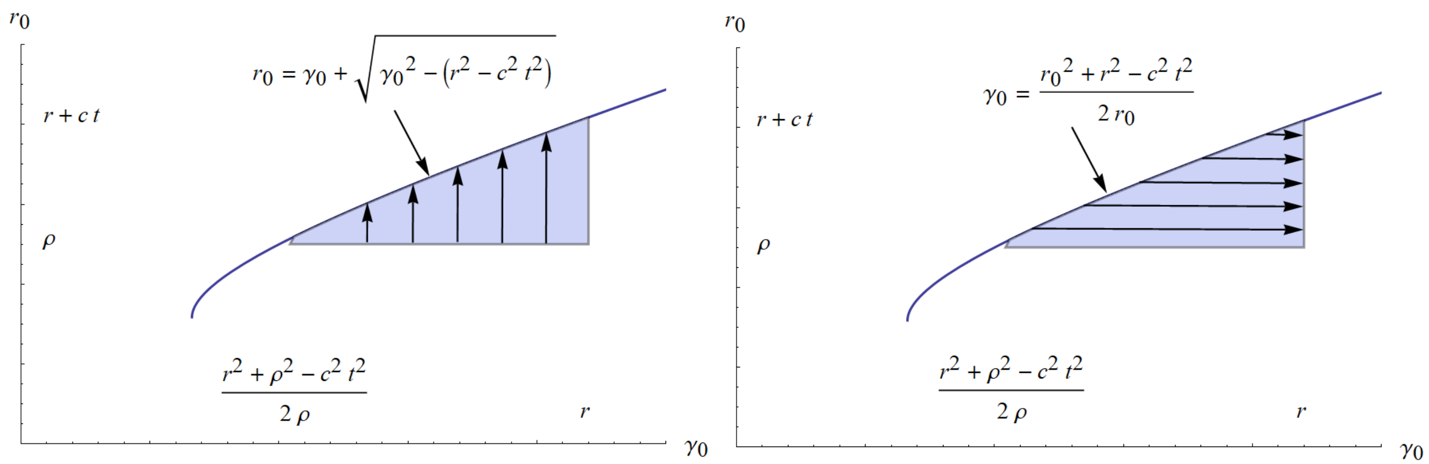
Consequently,

$$u(x,y,z,t) = \frac{A}{2c^2}\left[\int_r^{r\cos\gamma} \int_{\gamma_0 - \sqrt{\gamma_0^2 - (r^2-c^2t^2)}}^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2-c^2t^2)}} \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \left(-\frac{d\gamma_0}{r}\right) \right.$$
$$\left. - \int_r^{r\cos\beta} \int_\rho^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2-c^2t^2)}} \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \left(-\frac{d\gamma_0}{r}\right)\right]$$
$$= \frac{A}{2c^2 r}\left[\int_{r\cos\gamma}^r \int_{\gamma_0 - \sqrt{\gamma_0^2 - (r^2-c^2t^2)}}^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2-c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \right.$$
$$\left. - \int_{r\cos\beta}^r \int_\rho^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2-c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right]$$
$$= \frac{A}{2c^2 r}\left[\int_{\sqrt{r^2-c^2t^2}}^r \int_{\gamma_0 - \sqrt{\gamma_0^2 - (r^2-c^2t^2)}}^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2-c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \right.$$
$$\left. - \int_{\frac{r^2+\rho^2-c^2t^2}{2\rho}}^r \int_\rho^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2-c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right].$$

Now we will switch the order of integration in both integrals. The current mode of integration in the $\gamma_0 r_0$-plane for the first one is shown on the left.



Integrate over the domain as shown on the right to switch the order. Suppose that $r^2 < c^2t^2 + \rho^2$. Then the current mode of integration for the second integral in the $\gamma_0 r_0$-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order.

$$
\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2r}\left[\int_{r-ct}^{r+ct}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r}\frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} - \int_{\rho}^{r+ct}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r}\frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\right] \\
&= \frac{A}{2c^2r}\left[\int_{r-ct}^{r+ct}\left(\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r}\frac{r_0^2\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\right)dr_0 + \int_{r+ct}^{\rho}\left(\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r}\frac{r_0^2\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\right)dr_0\right] \\
&= \frac{A}{2c^2r}\int_{r-ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r}\frac{r_0^2\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\,dr_0
\end{aligned}
$$

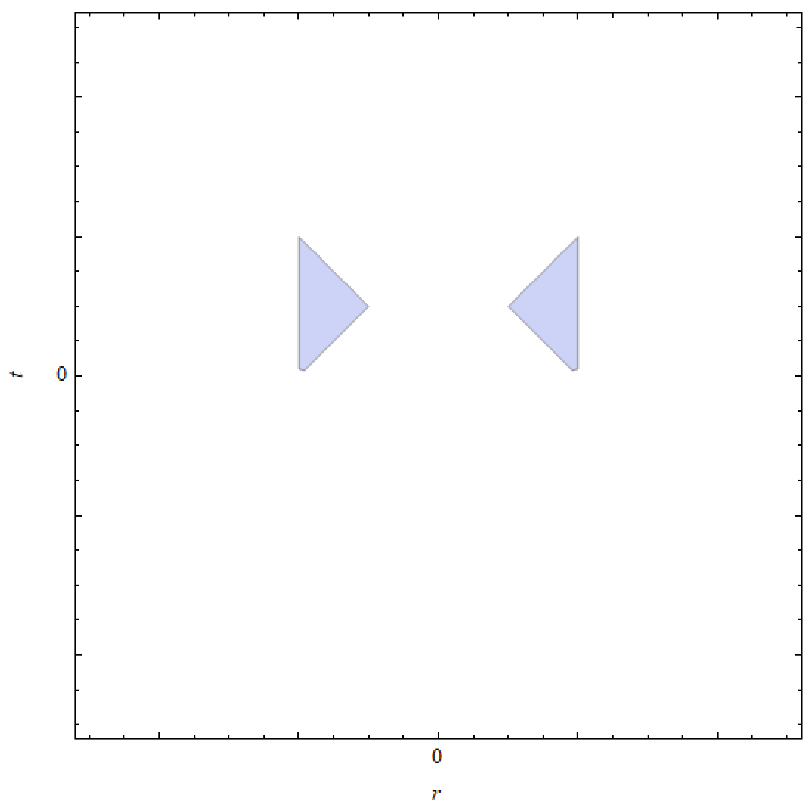
Make the following substitution.

$$
\begin{aligned}
s_0 &= r_0^2 - 2r_0\gamma_0 + r^2 \\
ds_0 &= -2r_0\,d\gamma_0 \quad \rightarrow \quad -\frac{ds_0}{2} = r_0\,d\gamma_0
\end{aligned}
$$

Therefore,

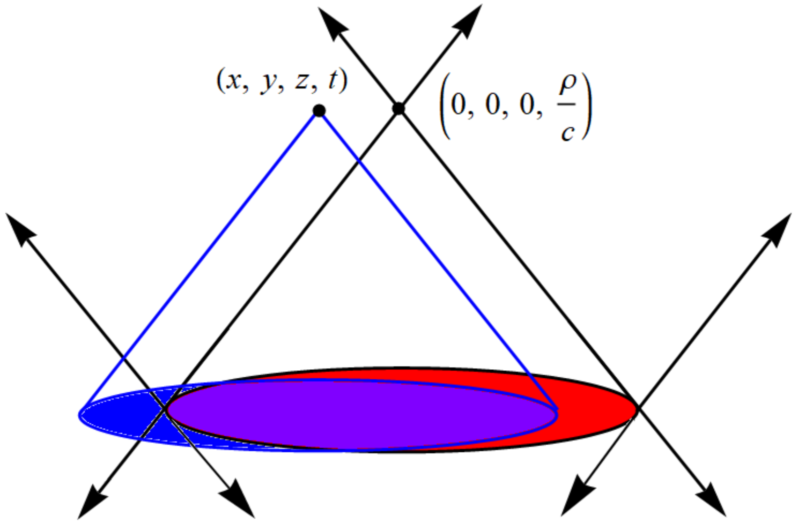
$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2 r}\int_{r-ct}^{\rho} r_0\left(\int_{c^2t^2}^{r_0^2-2r_0r+r^2}\frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\\
&= \frac{A}{2c^2 r}\int_{r-ct}^{\rho} r_0\left(\sqrt{c^2t^2}-\sqrt{(r_0-r)^2}\right)dr_0\\
&= \frac{A}{2c^2 r}\int_{r-ct}^{\rho} r_0\left(ct-|r_0-r|\right)dr_0\\
&= \frac{A}{2c^2 r}\left\{\int_{r-ct}^{r} r_0[ct-(r-r_0)]\,dr_0+\int_{r}^{\rho} r_0[ct-(r_0-r)]\,dr_0\right\}\\
&= \frac{A}{2c^2 r}\left\{\frac{1}{6}[-r^3-c^3t^3-2\rho^3+3ct(\rho^2-r^2)+3r(\rho^2+c^2t^2)]\right\}\\
&= \frac{A}{12c^2 r}\left[-r^3-c^3t^3-2\rho^3+3ct(\rho^2-r^2)+3r(\rho^2+c^2t^2)\right].
\end{aligned}$$

This solution is valid for $|\rho-ct|<r<\rho+ct$ and $r<\rho$ and $r-ct>0$ and $r^2<c^2t^2+\rho^2$. There's no need to consider the alternative case of $r^2>c^2t^2+\rho^2$ because there are no points in space-time that satisfy $|\rho-ct|<r<\rho+ct$ and $r<\rho$ and $r-ct>0$ and $r^2>c^2t^2+\rho^2$. Having said that, the preceding condition of validity can be relaxed to $|\rho-ct|<r<\rho+ct$ and $r<\rho$ and $r-ct>0$. A rough sketch of it in the rt-plane is shown below.

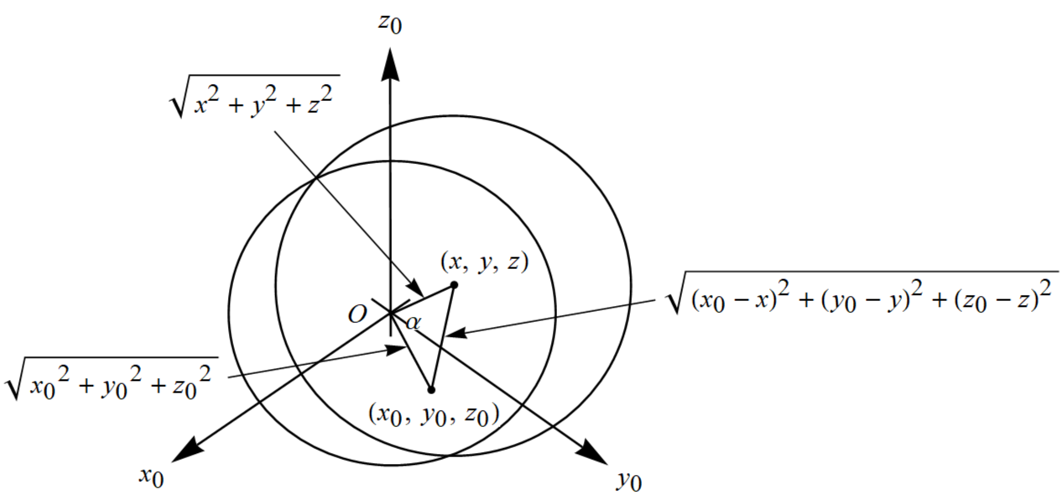


The Cyan Region - Position 4 of 4: $r < \rho$ and $r - ct < 0$

Several different intersections can occur in the cyan region. In this fourth part we will consider a point in space-time (x, y, z, t) inside the cylinder $r = \rho$ and above the cone $r - ct = 0$.



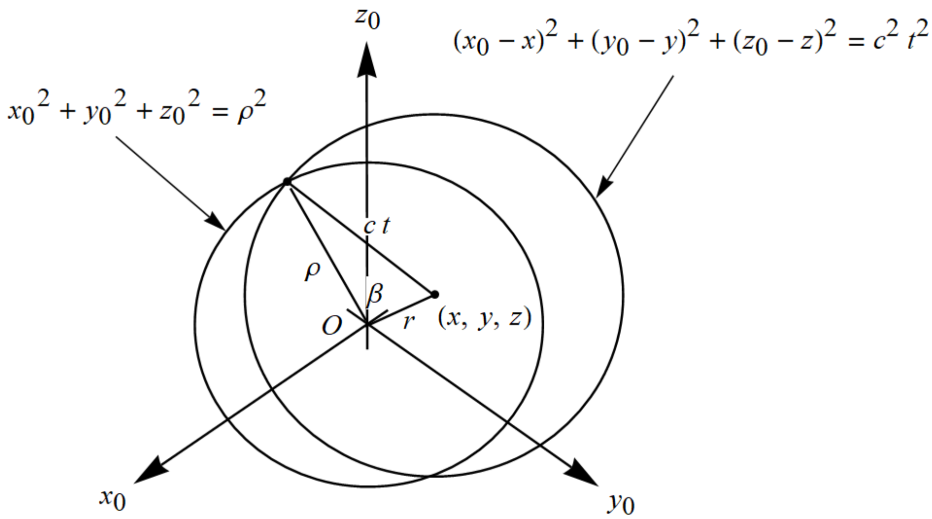
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves and the angle β at which one curve changes to another.



Use the law of cosines to find β.

$$c^2t^2 = r^2 + \rho^2 - 2\rho r\cos\beta \quad \rightarrow \quad \cos\beta = \frac{r^2+\rho^2-c^2t^2}{2\rho r}$$

The bounding curves are

$$\begin{aligned} x_0^2+y_0^2+z_0^2 &= \rho^2 & (x_0-x)^2+(y_0-y)^2+(z_0-z)^2 &= c^2t^2 \\ r_0^2 &= \rho^2 & r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0 &= \rho & r_0^2 - (2r\cos\alpha)r_0 - (c^2t^2-r^2) &= 0 \\ & & r_0 = \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha + 4(c^2t^2-r^2)}}{2} \\ & & r_0 = r\cos\alpha \pm \sqrt{r^2\cos^2\alpha + (c^2t^2-r^2)} \\ & & r_0 = r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2-r^2)} \end{aligned}$$

The positive sign was chosen because r_0, the distance to (x_0, y_0, z_0), must be positive. As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2}\iiint_{Q\cap T} \frac{dV_0}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}} \\ &= \frac{A}{4\pi c^2}\left[\int_0^\beta \int_0^{2\pi}\int_0^\rho \frac{r_0^2\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0}{\sqrt{r_0^2-2rr_0\cos\alpha+r^2}}\right. \\ &\qquad \left. + \int_\beta^\pi \int_0^{2\pi}\int_0^{r\cos\alpha+\sqrt{r^2\cos^2\alpha+(c^2t^2-r^2)}} \frac{r_0^2\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0}{\sqrt{r_0^2-2rr_0\cos\alpha+r^2}}\right]. \end{aligned}$$

Orienting the polar axis as we have makes it so that $\alpha = \phi_0$.

$$u(x,y,z,t) = \frac{A}{4\pi c^2}\left[\int_0^\beta \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \int_\beta^\pi \int_0^{2\pi} \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right]$$

Proceed to evaluate the integral in $d\theta_0$.

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2}\left[\left(\int_0^{2\pi} d\theta_0\right)\int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \left(\int_0^{2\pi} d\theta_0\right)\int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right] \\ &= \frac{A}{2c^2}\left[\int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right] \end{aligned}$$

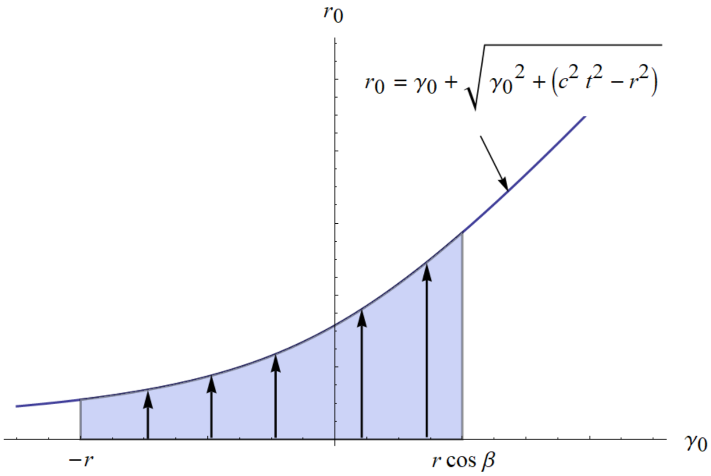
Make the following substitution in both integrals.

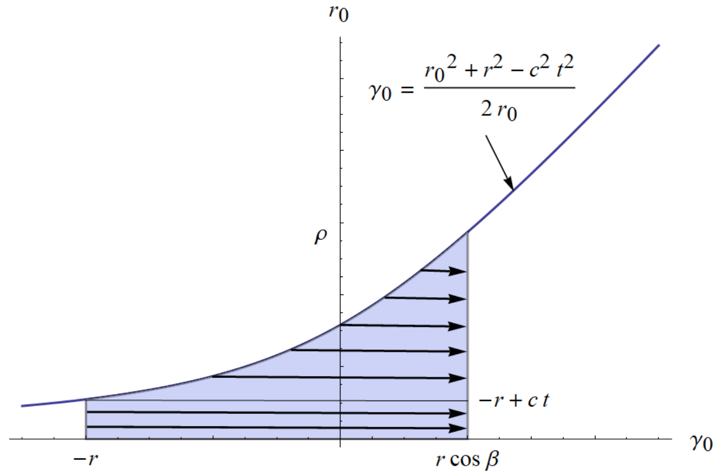
$$\begin{aligned} \gamma_0 &= r\cos\phi_0 \\ d\gamma_0 &= -r\sin\phi_0\, d\phi_0 \quad \rightarrow \quad -\frac{d\gamma_0}{r} = \sin\phi_0\, d\phi_0 \end{aligned}$$

Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{2c^2}\left[\int_r^{r\cos\beta} \int_0^\rho \frac{r_0^2\, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\left(-\frac{d\gamma_0}{r}\right) + \int_{r\cos\beta}^{-r} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2\, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\left(-\frac{d\gamma_0}{r}\right)\right] \\ &= \frac{A}{2c^2 r}\left[\int_{r\cos\beta}^r \int_0^\rho \frac{r_0^2\, dr_0\, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} + \int_{-r}^{r\cos\beta} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2\, dr_0\, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right]. \end{aligned}$$

Suppose that $r^2 < c^2t^2 + \rho^2$. Then the current mode of integration for the second integral is shown below on the left.





Integrate over the domain as shown on the right to switch the order of integration.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left(\int_0^\rho \int_{r\cos\beta}^{r} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} + \int_0^{-r+ct}\int_{-r}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right.$$
$$\left. + \int_{-r+ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right)$$

Make the following substitution in all three integrals.

$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$
$$ds_0 = -2r_0\,d\gamma_0 \quad \to \quad -\frac{ds_0}{2} = r_0\,d\gamma_0$$

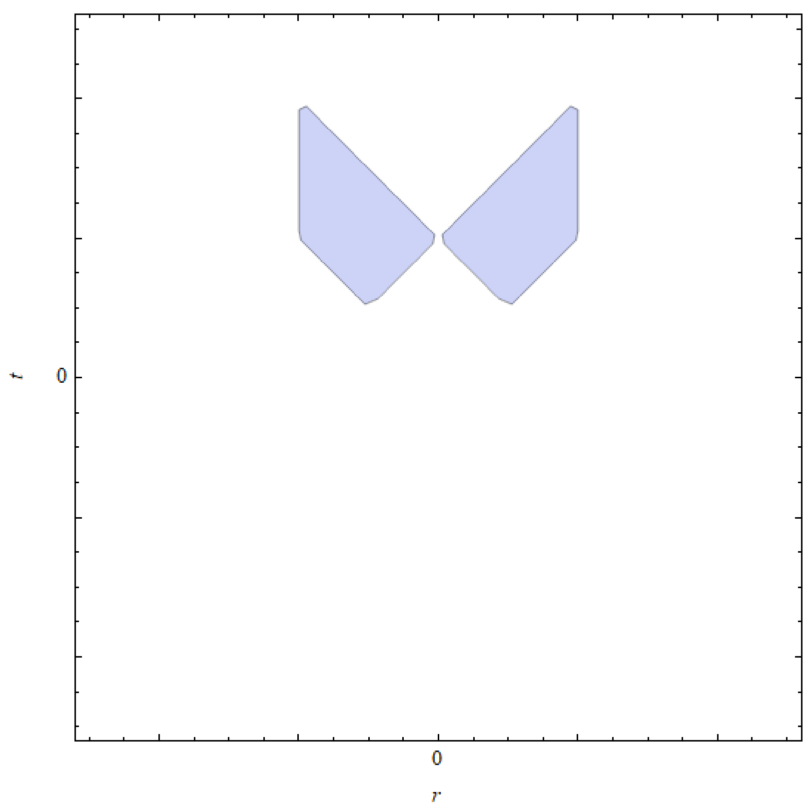
Consequently,

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\int_{r_0^2-2r_0r\cos\beta+r^2}^{r_0^2-2r_0r+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0 + \int_0^{-r+ct} r_0\left(\int_{r_0^2+2r_0r+r^2}^{r_0^2-2r_0r\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\right.$$
$$\left. + \int_{-r+ct}^{\rho} r_0\left(\int_{c^2t^2}^{r_0^2-2r_0r\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\right]$$
$$= \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\sqrt{r_0^2 - 2r_0r\cos\beta + r^2} - |r_0 - r|\right)dr_0 + \int_0^{-r+ct} r_0\left(r_0 + r - \sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\right)dr_0\right.$$
$$\left. + \int_{-r+ct}^{\rho} r_0\left(ct - \sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\right)dr_0\right].$$

Therefore,

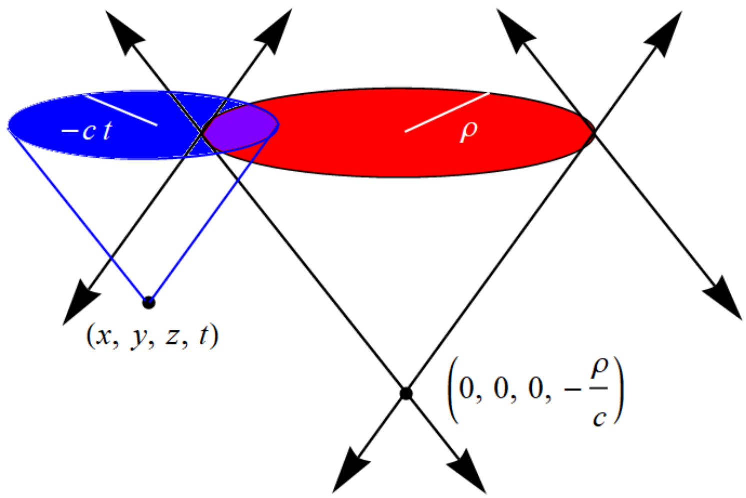
$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2r}\left[\int_0^\rho r_0\sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\,dr_0 - \int_0^\rho r_0|r_0 - r|\,dr_0\right.\\
&\qquad + \int_0^{-r+ct} r_0(r_0+r)\,dr_0 - \int_0^{-r+ct} r_0\sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\,dr_0\\
&\qquad\qquad \left. + \int_{-r+ct}^\rho r_0ct\,dr_0 - \int_{-r+ct}^\rho r_0\sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\,dr_0\right]\\
&= \frac{A}{2c^2r}\left[-\int_0^\rho r_0|r_0 - r|\,dr_0 + \int_0^{-r+ct} r_0(r_0+r)\,dr_0 + \int_{-r+ct}^\rho r_0ct\,dr_0\right]\\
&= \frac{A}{2c^2r}\left[-\int_0^r r_0(r-r_0)\,dr_0 - \int_r^\rho r_0(r_0-r)\,dr_0 + \int_0^{-r+ct} r_0(r_0+r)\,dr_0 + \int_{-r+ct}^\rho r_0ct\,dr_0\right]\\
&= \frac{A}{2c^2r}\left\{\frac{1}{6}[-r^3 - c^3t^3 - 2\rho^3 + 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)]\right\}\\
&= \frac{A}{12c^2r}\left[-r^3 - c^3t^3 - 2\rho^3 + 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right].
\end{aligned}$$

This solution is valid for $|\rho - ct| < r < \rho + ct$ and $r < \rho$ and $r - ct < 0$ and $r^2 < c^2t^2 + \rho^2$. There's no need to consider the alternative case of $r^2 > c^2t^2 + \rho^2$ because there are no points in space-time that satisfy $|\rho - ct| < r < \rho + ct$ and $r < \rho$ and $r - ct < 0$ and $r^2 > c^2t^2 + \rho^2$. Having said that, the preceding condition of validity can be relaxed to $|\rho - ct| < r < \rho + ct$ and $r < \rho$ and $r - ct < 0$. A rough sketch of it in the rt-plane is shown below.

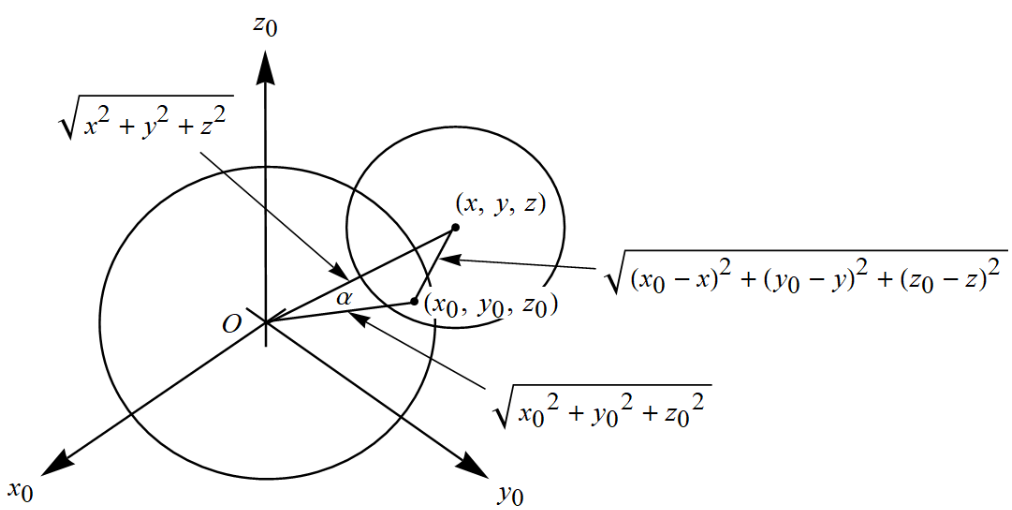


The Magenta Region - Position 1 of 4: $r > \rho$ and $r + ct > 0$

Several different intersections can occur in the magenta region. In this first part we will consider a point in space-time (x, y, z, t) outside the cylinder $r = \rho$ and above the cone $r + ct = 0$.



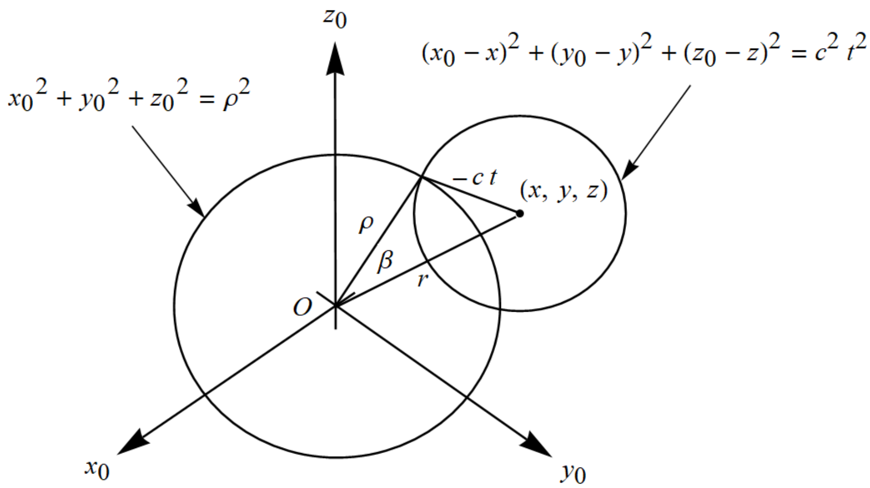
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves of the balls and the maximum polar angle β.



Use the law of cosines to find β.

$$c^2t^2 = r^2 + \rho^2 - 2\rho r\cos\beta \quad \rightarrow \quad \cos\beta = \frac{r^2 + \rho^2 - c^2t^2}{2\rho r}$$

The bounding curves are as follows.

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= \rho^2 \\ r_0^2 &= \rho^2 \\ r_0 &= \rho \end{aligned} \qquad \begin{aligned} (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= c^2t^2 \\ r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0^2 - (2r\cos\alpha)r_0 + (r^2 - c^2t^2) &= 0 \\ r_0 &= \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha - 4(r^2 - c^2t^2)}}{2} \\ r_0 &= r\cos\alpha \pm \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)} \\ r_0 &= r\cos\alpha - \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)} \end{aligned}$$

The minus sign was chosen because we want the part of the bounding curve closest to O. As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x, y, z, t) &= \frac{A}{4\pi c^2} \iiint_{Q\cap T} \frac{dV_0}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}} \\ &= \frac{A}{4\pi c^2} \int_0^\beta \int_0^{2\pi} \int_{r\cos\alpha - \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}}^{\rho} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}. \end{aligned}$$

Orienting the polar axis as we have makes it so that $\alpha = \phi_0$.

$$u(x, y, z, t) = \frac{A}{4\pi c^2} \int_0^\beta \int_0^{2\pi} \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^{\rho} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}$$

Proceed to evaluate the integral in $d\theta_0$.

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{4\pi c^2}\left(\int_0^{2\pi} d\theta_0\right)\int_0^{\beta}\int_{r\cos\phi_0-\sqrt{r^2\cos^2\phi_0-(r^2-c^2t^2)}}^{\rho} \frac{r_0^2\sin\phi_0\,dr_0\,d\phi_0}{\sqrt{r_0^2-2rr_0\cos\phi_0+r^2}} \\
&= \frac{A}{2c^2}\int_0^{\beta}\int_{r\cos\phi_0-\sqrt{r^2\cos^2\phi_0-(r^2-c^2t^2)}}^{\rho} \frac{r_0^2\sin\phi_0\,dr_0\,d\phi_0}{\sqrt{r_0^2-2rr_0\cos\phi_0+r^2}}
\end{aligned}$$

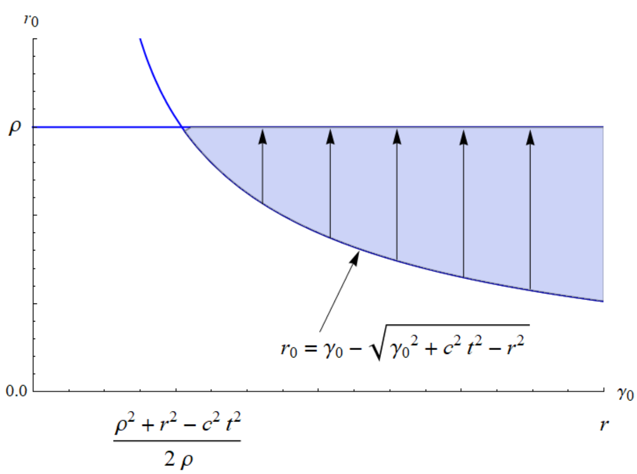
Make the following change of variables.

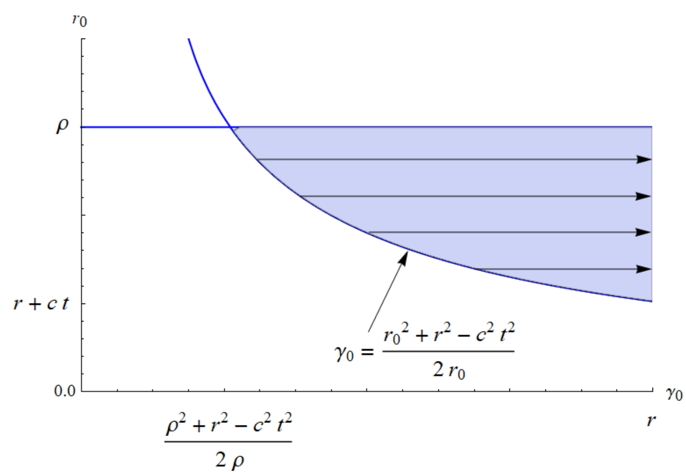
$$\begin{aligned}
\gamma_0 &= r\cos\phi_0 \\
d\gamma_0 &= -r\sin\phi_0\,d\phi_0 \quad \rightarrow \quad -\frac{d\gamma_0}{r} = \sin\phi_0\,d\phi_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2}\int_r^{r\cos\beta}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\rho} \frac{r_0^2\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\left(-\frac{d\gamma_0}{r}\right) \\
&= \frac{A}{2c^2r}\int_{r\cos\beta}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\rho} \frac{r_0^2\,dr_0\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} \\
&= \frac{A}{2c^2r}\int_{\frac{\rho^2+r^2-c^2t^2}{2\rho}}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\rho} \frac{r_0^2\,dr_0\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}. \qquad (2)
\end{aligned}$$

Suppose first that $r^2 > c^2t^2+\rho^2$. Then the current mode of integration in the $\gamma_0 r_0$-plane is shown on the left.





Integrate over the domain as shown on the right in order to switch the order of integration.

$$u(x,y,z,t) = \frac{A}{2c^2r}\int_{r+ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}$$

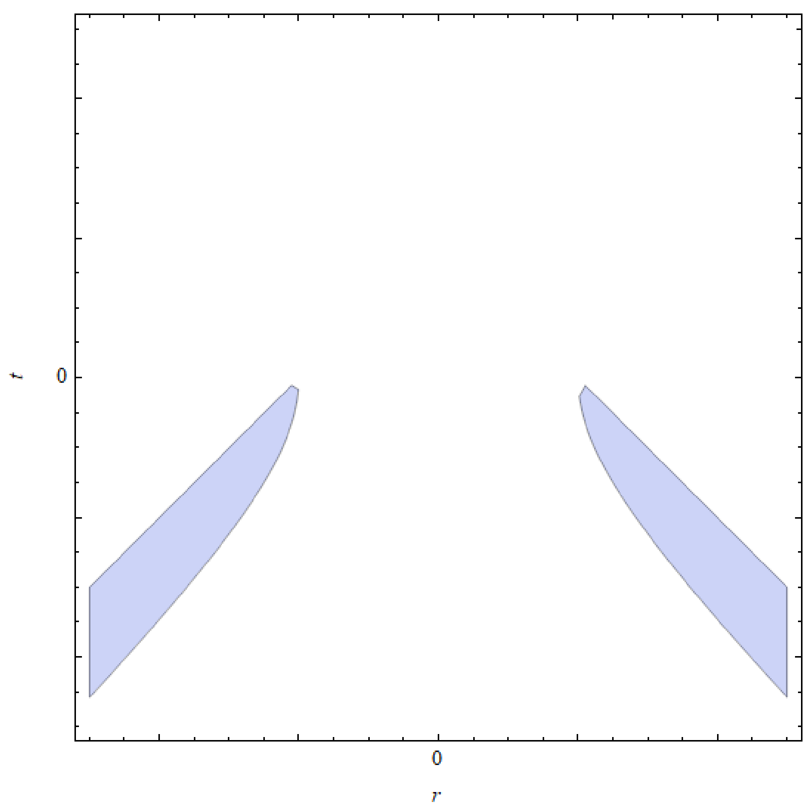
Make the following substitution.

$$\begin{aligned}
s_0 &= r_0^2-2r_0\gamma_0+r^2 \\
ds_0 &= -2r_0\,d\gamma_0 \quad \rightarrow \quad -\frac{ds_0}{2} = r_0\,d\gamma_0
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0\left(\int_{c^2t^2}^{r_0^2-2rr_0+r^2}\frac{-ds_0}{2\sqrt{s_0}}\right)dr_0 \\
&= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0\left[\int_{(r_0-r)^2}^{c^2t^2}\frac{ds_0}{2\sqrt{s_0}}\right]dr_0 \\
&= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0\left[\sqrt{c^2t^2}-\sqrt{(r_0-r)^2}\right]dr_0 \\
&= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0\left(-ct-|r_0-r|\right)dr_0 \\
&= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0[-ct-(r-r_0)]\,dr_0 \\
&= \frac{A}{2c^2r}\left\{\frac{1}{6}\left[r^3+c^3t^3+2\rho^3+3ct(r^2-\rho^2)+3r(c^2t^2-\rho^2)\right]\right\} \\
&= \frac{A}{12c^2r}\left[r^3+c^3t^3+2\rho^3+3ct(r^2-\rho^2)+3r(c^2t^2-\rho^2)\right].
\end{aligned}$$

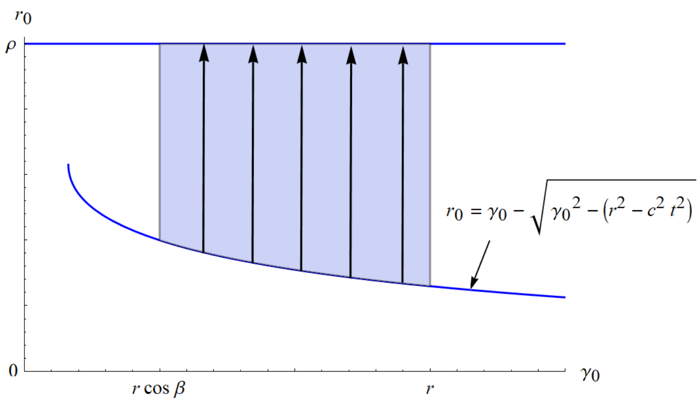
This solution is valid for $|\rho+ct|<r<\rho-ct$ and $r>\rho$ and $r+ct>0$ and $r^2>c^2t^2+\rho^2$. A rough sketch of the validity condition in the rt-plane is shown below.

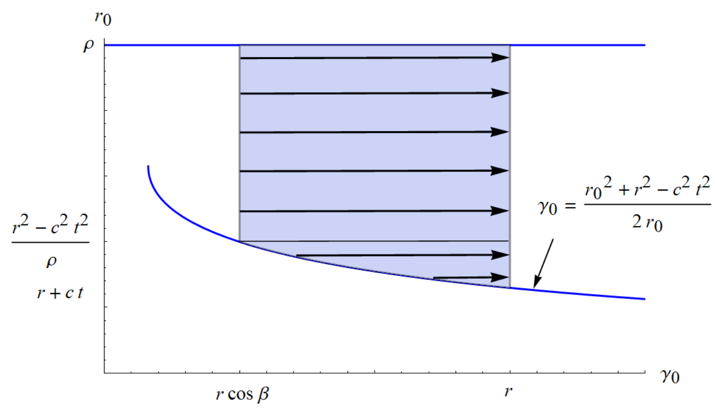


Return to equation (2).

$$u(x,y,z,t) = \frac{A}{2c^2r}\int_{\frac{\rho^2+r^2-c^2t^2}{2\rho}}^{r}\int_{\gamma_0-\sqrt{\gamma_0^2-(r^2-c^2t^2)}}^{\rho}\frac{r_0^2\,dr_0\,d\gamma_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}} \tag{2}$$

Suppose secondly that $r^2 < c^2t^2+\rho^2$. Then the current mode of integration in the $\gamma_0 r_0$-plane is shown on the left.





Integrate over the domain as shown on the right in order to switch the order of integration.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_{r+ct}^{\frac{r^2-c^2t^2}{\rho}}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r}\frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho}\int_{r\cos\beta}^{r}\frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2-2r_0\gamma_0+r^2}}\right]$$

Make the following substitution.

$$\begin{aligned} s_0 &= r_0^2-2r_0\gamma_0+r^2 \\ ds_0 &= -2r_0\,d\gamma_0 \quad\to\quad -\frac{ds_0}{2} = r_0\,d\gamma_0 \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{2c^2r}\left[\int_{r+ct}^{\frac{r^2-c^2t^2}{\rho}} r_0\left(\int_{c^2t^2}^{r_0^2-2rr_0+r^2}\frac{-ds_0}{2\sqrt{s_0}}\right)dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\left(\int_{r_0^2-2r_0r\cos\beta+r^2}^{r_0^2-2rr_0+r^2}\frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\right] \\ &= \frac{A}{2c^2r}\left[\int_{r+ct}^{\frac{r^2-c^2t^2}{\rho}} r_0\left(-ct-|r_0-r|\right)dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\left(\sqrt{r_0^2-2r_0r\cos\beta+r^2}-|r_0-r|\right)dr_0\right] \\ &= \frac{A}{2c^2r}\left[-\int_{r+ct}^{\frac{r^2-c^2t^2}{\rho}} r_0ct\,dr_0-\int_{r+ct}^{\rho} r_0|r_0-r|\,dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2-2r_0r\cos\beta+r^2}\,dr_0\right] \\ &= \frac{A}{2c^2r}\left[-\int_{r+ct}^{\frac{r^2-c^2t^2}{\rho}} r_0ct\,dr_0-\int_{r+ct}^{\rho} r_0(r-r_0)\,dr_0+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2-2r_0r\cos\beta+r^2}\,dr_0\right] \\ &= \frac{A}{2c^2r}\left[\frac{-3ct(r^2-c^2t^2)^2+(r+ct)^3\rho^2-3r\rho^4+2\rho^5}{6\rho^2}+\int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2-2r_0r\cos\beta+r^2}\,dr_0\right]. \end{aligned}$$

Proof for the result of this final integral will be given at the end.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left\{\frac{-3ct(r^2-c^2t^2)^2+(r+ct)^3\rho^2-3r\rho^4+2\rho^5}{6\rho^2}\right.$$

$$+\frac{(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]}{8\rho^3}\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}$$

$$\left.-\frac{ct[\rho^4-(r^2-c^2t^2)^2]}{4\rho^2}\right\}$$

$$= \frac{A}{2c^2r}\left\{\frac{(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]}{8\rho^3}\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}\right.$$

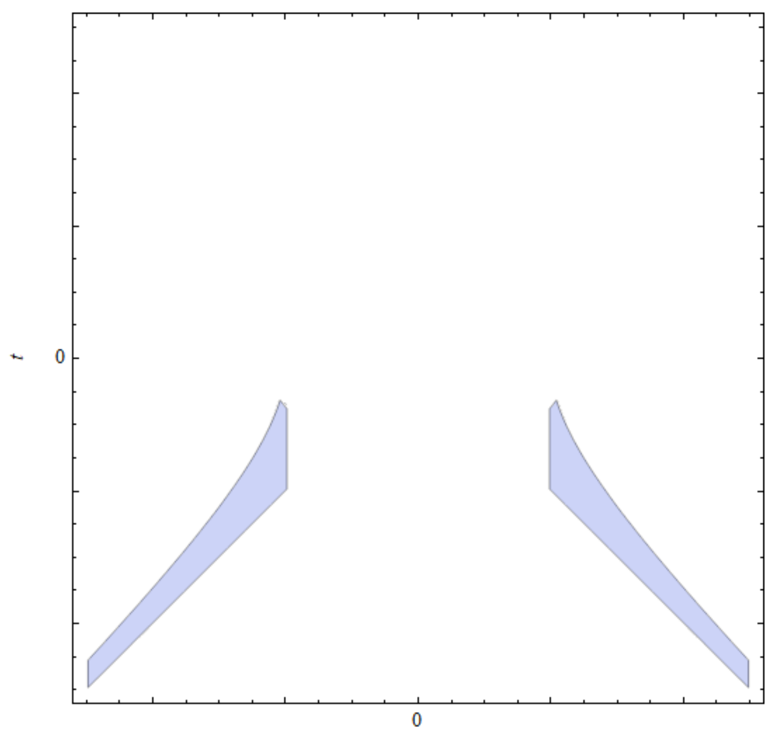
$$\left.+\frac{-3ct(r^2-c^2t^2)^2+2\rho^2(r+ct)^3-3\rho^4(2r+ct)+4\rho^5}{12\rho^2}\right\}$$

Therefore,

$$u(x,y,z,t) = \frac{A}{48\rho^3c^2r}\left\{3(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}\right.$$

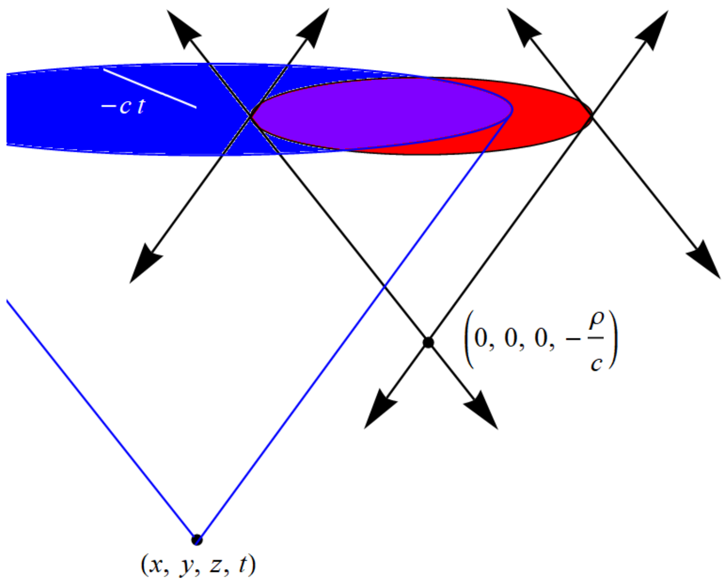
$$\left.+2\rho\left[-3ct(r^2-c^2t^2)^2+2\rho^2(r+ct)^3-3\rho^4(2r+ct)+4\rho^5\right]\right\}.$$

This solution is valid for $|\rho+ct|<r<\rho-ct$ and $r>\rho$ and $r+ct>0$ and $r^2<c^2t^2+\rho^2$. A rough sketch of the validity condition in the rt-plane is shown below.

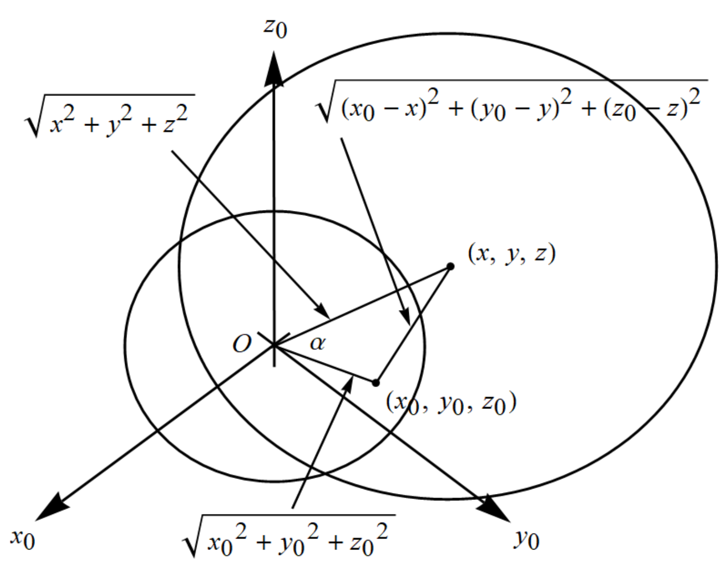


The Magenta Region - Position 2 of 4: $r > \rho$ and $r + ct < 0$

Several different intersections can occur in the magenta region. In this second part we will consider a point in space-time (x, y, z, t) outside the cylinder $r = \rho$ and below the cone $r + ct = 0$.



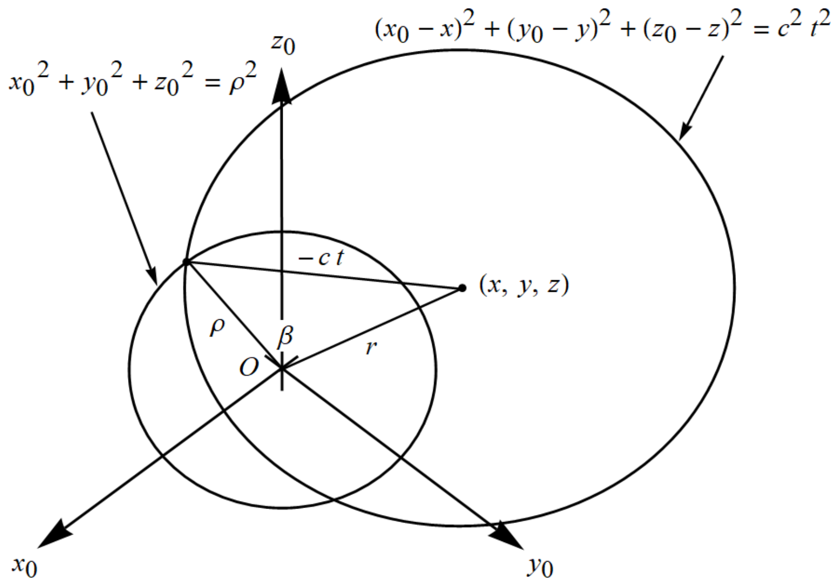
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves and the angle β at which one curve changes to the other.



Use the law of cosines again to find β.

$$c^2t^2 = \rho^2 + r^2 - 2\rho r \cos\beta \quad \rightarrow \quad \cos\beta = \frac{\rho^2 + r^2 - c^2t^2}{2\rho r}$$

The bounding curves are

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= \rho^2 \\ r_0^2 &= \rho^2 \\ r_0 &= \rho \end{aligned} \qquad \begin{aligned} (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= c^2t^2 \\ r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0^2 - (2r\cos\alpha)r_0 - (c^2t^2 - r^2) &= 0 \\ r_0 &= \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha + 4(c^2t^2 - r^2)}}{2} \\ r_0 &= r\cos\alpha \pm \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)} \\ r_0 &= r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)}. \end{aligned}$$

The positive sign is chosen because r_0, the distance to (x_0, y_0, z_0), must be positive. As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2} \iiint_{Q\cap T} \frac{dV_0}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}} \\ &= \frac{A}{4\pi c^2} \left[\int_0^\beta \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} \right. \\ &\qquad \left. + \int_\beta^\pi \int_0^{2\pi} \int_0^{r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} \right]. \end{aligned}$$

Orienting the polar axis in the direction of (x, y, z) makes it so that $\alpha = \phi_0$.

$$u(x,y,z,t) = \frac{A}{4\pi c^2}\left[\int_0^\beta \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \int_\beta^\pi \int_0^{2\pi} \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right]$$

Evaluate the integral in $d\theta_0$.

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2}\left[\left(\int_0^{2\pi} d\theta_0\right) \int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \left(\int_0^{2\pi} d\theta_0\right) \int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right] \\ &= \frac{A}{2c^2}\left[\int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right] \end{aligned}$$

Make the following substitution in both integrals.

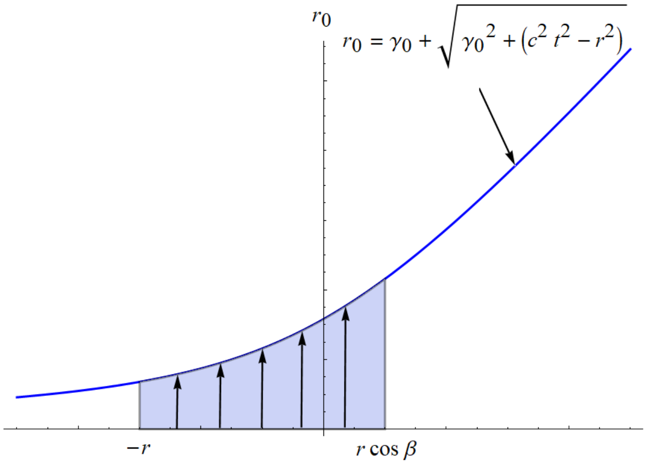
$$\begin{aligned} \gamma_0 &= r\cos\phi_0 \\ d\gamma_0 &= -r\sin\phi_0 \, d\phi_0 \quad \rightarrow \quad -\frac{d\gamma_0}{r} = \sin\phi_0 \, d\phi_0 \end{aligned}$$

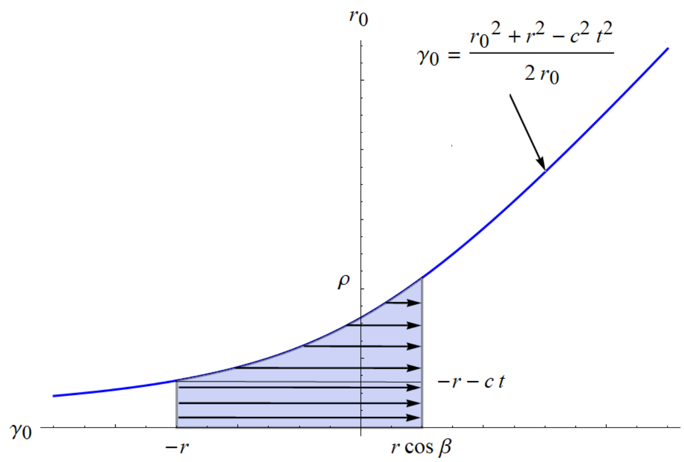
Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{2c^2}\left[\int_r^{r\cos\beta} \int_0^\rho \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\left(-\frac{d\gamma_0}{r}\right) + \int_{r\cos\beta}^{-r} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\left(-\frac{d\gamma_0}{r}\right)\right] \\ &= \frac{A}{2c^2 r}\left[\int_{r\cos\beta}^r \int_0^\rho \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} + \int_{-r}^{r\cos\beta} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right]. \end{aligned}$$

The order of integration in both integrals will now be switched.

Suppose that $r^2 < c^2t^2 + \rho^2$. Then the current mode of integration in the second integral is shown on the left.





Integrate over the domain as shown on the right to switch the order of the second integral.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_0^\rho \int_{r\cos\beta}^r \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} + \int_0^{-r-ct}\int_{-r}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \right.$$
$$\left. + \int_{-r-ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right]$$

Make the following substitution in all three integrals.

$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$
$$ds_0 = -2r_0\,d\gamma_0 \quad \rightarrow \quad -\frac{ds_0}{2} = r_0\,d\gamma_0$$

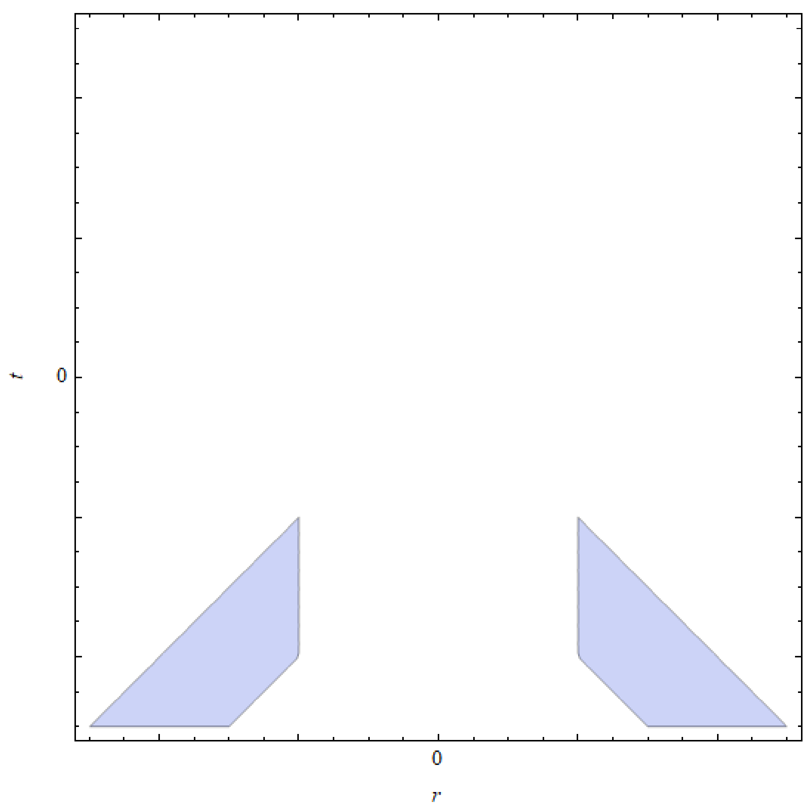
Consequently,

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\int_{r_0^2-2rr_0\cos\beta+r^2}^{r_0^2-2rr_0+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0 \right.$$
$$+ \int_0^{-r-ct} r_0\left(\int_{r_0^2+2rr_0+r^2}^{r_0^2-2rr_0\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0$$
$$\left. + \int_{-r-ct}^{\rho} r_0\left(\int_{c^2t^2}^{r_0^2-2rr_0\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\right]$$
$$= \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\sqrt{r_0^2 - 2rr_0\cos\beta + r^2} - |r_0 - r|\right)dr_0 \right.$$
$$+ \int_0^{-r-ct} r_0\left(r_0 + r - \sqrt{r_0^2 - 2rr_0\cos\beta + r^2}\right)dr_0$$
$$\left. + \int_{-r-ct}^{\rho} r_0\left(-ct - \sqrt{r_0^2 - 2rr_0\cos\beta + r^2}\right)dr_0\right].$$

Therefore,

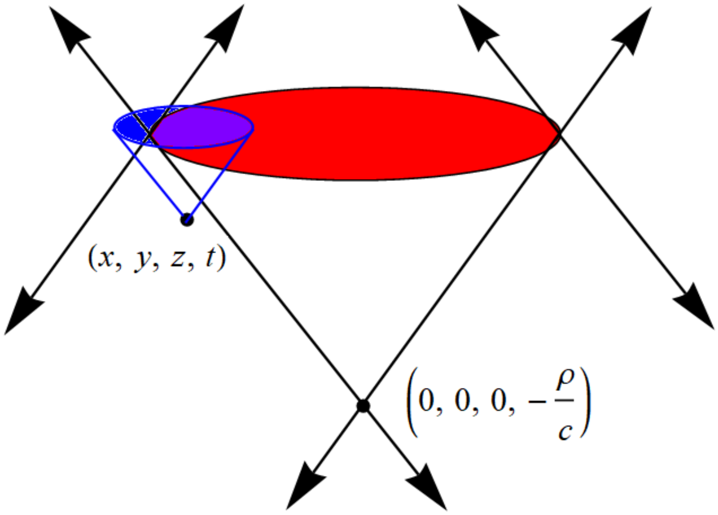
$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2r}\left[\int_0^{\rho} r_0\sqrt{r_0^2-2rr_0\cos\beta+r^2}\,dr_0 - \int_0^{\rho} r_0|r_0-r|\,dr_0 + \int_0^{-r-ct} r_0(r_0+r)\,dr_0\right.\\
&\qquad - \int_0^{-r-ct} r_0\sqrt{r_0^2-2rr_0\cos\beta+r^2}\,dr_0 - \int_{-r-ct}^{\rho} r_0ct\,dr_0\\
&\qquad \left. - \int_{-r-ct}^{\rho} r_0\sqrt{r_0^2-2rr_0\cos\beta+r^2}\,dr_0\right]\\
&= \frac{A}{2c^2r}\left[-\int_0^{\rho} r_0|r_0-r|\,dr_0 + \int_0^{-r-ct} r_0(r_0+r)\,dr_0 - \int_{-r-ct}^{\rho} r_0ct\,dr_0\right]\\
&= \frac{A}{2c^2r}\left[-\int_0^{\rho} r_0(r-r_0)\,dr_0 + \int_0^{-r-ct} r_0(r_0+r)\,dr_0 - \int_{-r-ct}^{\rho} r_0ct\,dr_0\right]\\
&= \frac{A}{2c^2r}\left\{\frac{1}{6}\left[r^3+c^3t^3+2\rho^3+3ct(r^2-\rho^2)+3r(c^2t^2-\rho^2)\right]\right\}\\
&= \frac{A}{12c^2r}\left[r^3+c^3t^3+2\rho^3+3ct(r^2-\rho^2)+3r(c^2t^2-\rho^2)\right].
\end{aligned}$$

This solution is valid for $|\rho+ct|<r<\rho-ct$ and $r>\rho$ and $r+ct<0$ and $r^2<c^2t^2+\rho^2$. There's no need to consider the alternative case of $r^2>c^2t^2+\rho^2$ because there are no points in space-time that satisfy $|\rho+ct|<r<\rho-ct$ and $r>\rho$ and $r+ct<0$ and $r^2>c^2t^2+\rho^2$. Having said that, the preceding condition of validity can be relaxed to $|\rho+ct|<r<\rho-ct$ and $r>\rho$ and $r+ct<0$. A rough sketch of it in the rt-plane is shown below.

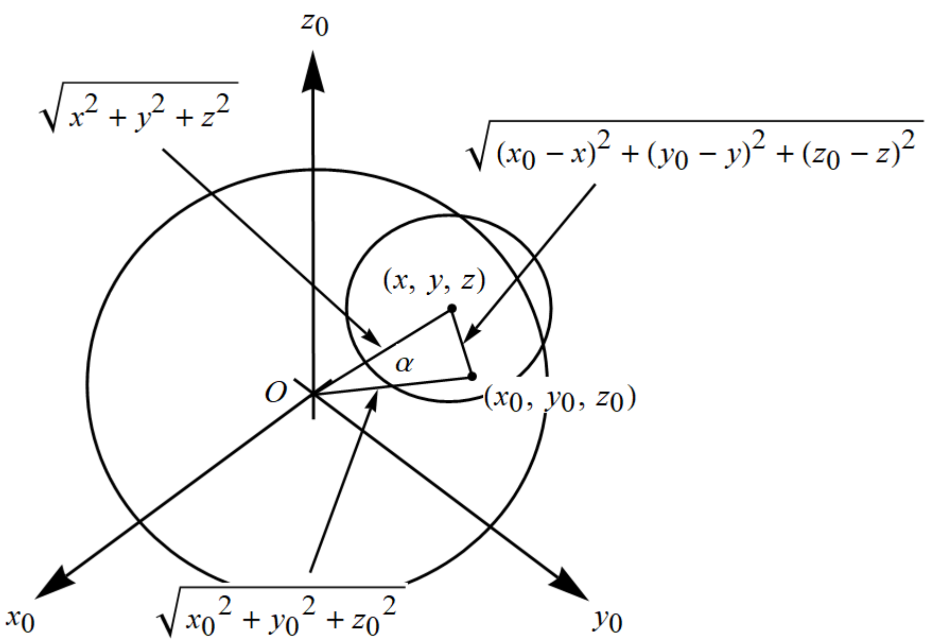


The Magenta Region - Position 3 of 4: $r < \rho$ and $r + ct > 0$

Several different intersections can occur in the magenta region. In this third part we will consider a point in space-time (x, y, z, t) inside the cylinder $r = \rho$ and above the cone $r + ct = 0$.



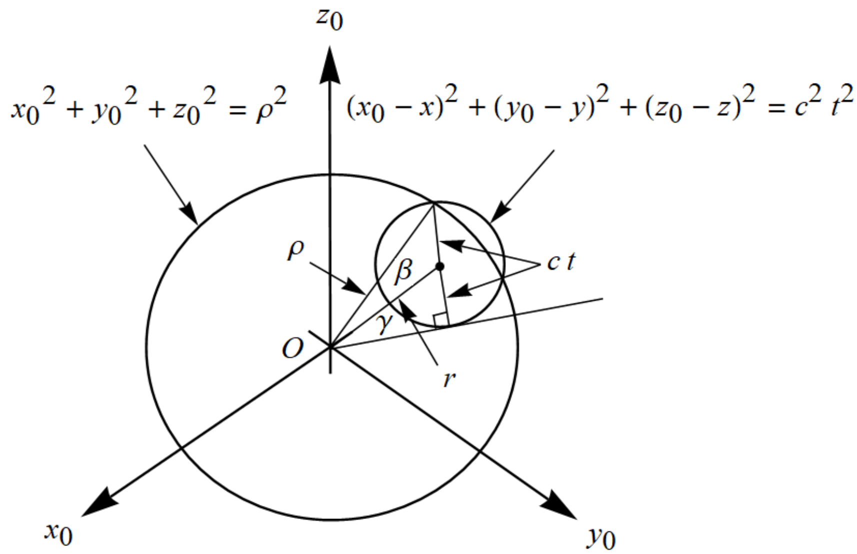
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves and the angles, β (the angle at which one curve changes to another) and γ (the maximum polar angle).



Use the law of cosines in the upper triangle to find β.

$$c^2t^2 = r^2 + \rho^2 - 2\rho r \cos\beta \quad \rightarrow \quad \cos\beta = \frac{r^2 + \rho^2 - c^2t^2}{2\rho r}$$

According to the Pythagorean theorem, the length of the missing side of the lower triangle is $r\cos\gamma = \sqrt{r^2 - c^2t^2}$. The bounding curves are

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= \rho^2 & \qquad (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= c^2t^2 \\ r_0^2 &= \rho^2 & r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0 &= \rho & r_0^2 - (2r\cos\alpha)r_0 + (r^2 - c^2t^2) &= 0 \\ & & r_0 &= \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha - 4(r^2 - c^2t^2)}}{2} \\ & & r_0 &= r\cos\alpha \pm \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}. \end{aligned}$$

As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2} \iiint_{Q\cap T} \frac{dV_0}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}} \\ &= \frac{A}{4\pi c^2}\left[\int_0^\gamma \int_0^{2\pi} \int_{r\cos\alpha - \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}}^{r\cos\alpha + \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}} \right. \\ &\qquad \left. - \int_0^\beta \int_0^{2\pi} \int_\rho^{r\cos\alpha + \sqrt{r^2\cos^2\alpha - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}\right]. \end{aligned}$$

What we're essentially doing is integrating over the whole ball of radius $-ct$ and then subtracting the volume integral over the part that lies outside the ball of radius ρ.

Orienting the polar axis as we have makes it so that $\alpha = \phi_0$.

$$\begin{aligned}u(x,y,z,t) = \frac{A}{4\pi c^2}\Bigg[&\int_0^\gamma \int_0^{2\pi} \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \\ &- \int_0^\beta \int_0^{2\pi} \int_\rho^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\Bigg]\end{aligned}$$

Proceed to evaluate the integral in $d\theta_0$.

$$\begin{aligned}u(x,y,z,t) &= \frac{A}{4\pi c^2}\Bigg[\left(\int_0^{2\pi} d\theta_0\right) \int_0^\gamma \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \\ &\qquad - \left(\int_0^{2\pi} d\theta_0\right) \int_0^\beta \int_\rho^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\Bigg] \\ &= \frac{A}{2c^2}\Bigg[\int_0^\gamma \int_{r\cos\phi_0 - \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}}^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} \\ &\qquad - \int_0^\beta \int_\rho^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 - (r^2 - c^2t^2)}} \frac{r_0^2 \sin\phi_0 \, dr_0 \, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\Bigg]\end{aligned}$$

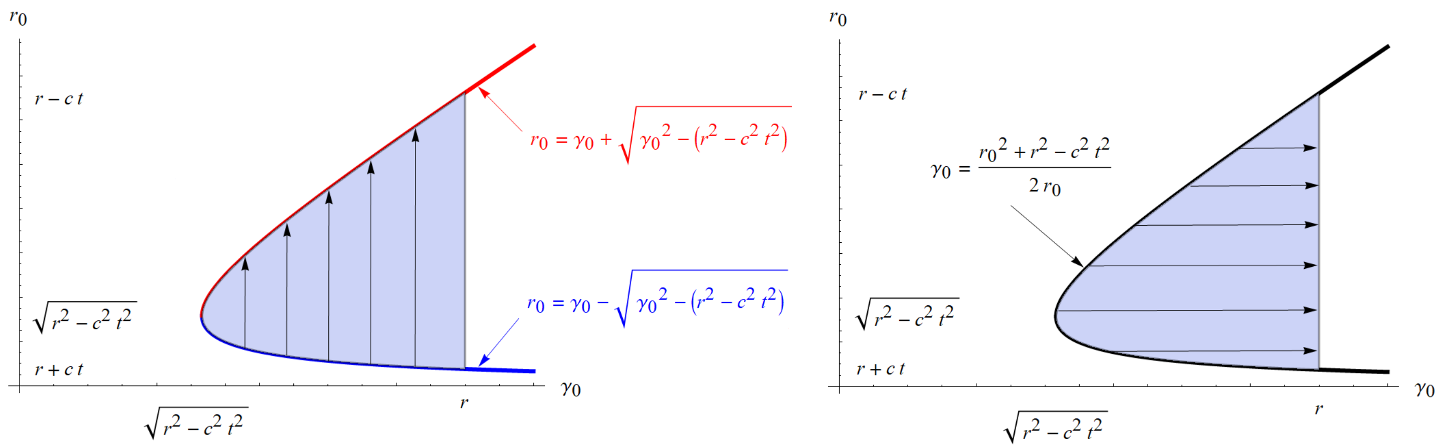
Make the following substitution in both integrals.

$$\begin{aligned}\gamma_0 &= r\cos\phi_0 \\ d\gamma_0 &= -r\sin\phi_0 \, d\phi_0 \quad \to \quad -\frac{d\gamma_0}{r} = \sin\phi_0 \, d\phi_0\end{aligned}$$

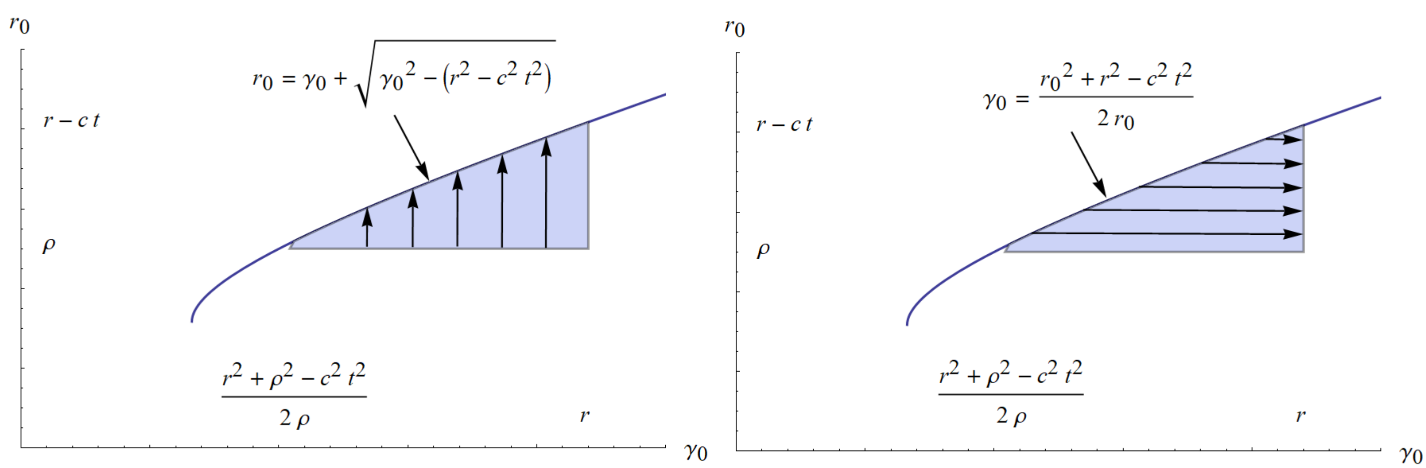
Consequently,

$$\begin{aligned}u(x,y,z,t) &= \frac{A}{2c^2}\Bigg[\int_r^{r\cos\gamma} \int_{\gamma_0 - \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}}^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}} \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \left(-\frac{d\gamma_0}{r}\right) \\ &\qquad - \int_r^{r\cos\beta} \int_\rho^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}} \frac{r_0^2 \, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \left(-\frac{d\gamma_0}{r}\right)\Bigg] \\ &= \frac{A}{2c^2 r}\Bigg[\int_{r\cos\gamma}^r \int_{\gamma_0 - \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}}^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \\ &\qquad - \int_{r\cos\beta}^r \int_\rho^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\Bigg] \\ &= \frac{A}{2c^2 r}\Bigg[\int_{\sqrt{r^2 - c^2t^2}}^r \int_{\gamma_0 - \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}}^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \\ &\qquad - \int_{\frac{r^2 + \rho^2 - c^2t^2}{2\rho}}^r \int_\rho^{\gamma_0 + \sqrt{\gamma_0^2 - (r^2 - c^2t^2)}} \frac{r_0^2 \, dr_0 \, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\Bigg].\end{aligned}$$

Now we will switch the order of integration in both integrals. The current mode of integration in the $\gamma_0 r_0$-plane for the first one is shown on the left.



Integrate over the domain as shown on the right to switch the order. Suppose that $r^2 < c^2t^2 + \rho^2$. Then the current mode of integration in the $\gamma_0 r_0$-plane of the second integral is shown below on the left.



Integrate over the domain as shown on the right to switch the order.

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2 r}\left[\int_{r+ct}^{r-ct}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} - \int_{\rho}^{r-ct}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right] \\
&= \frac{A}{2c^2 r}\left[\int_{r+ct}^{r-ct}\left(\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{r_0^2\,d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right)dr_0 + \int_{r-ct}^{\rho}\left(\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{r_0^2\,d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right)dr_0\right] \\
&= \frac{A}{2c^2 r}\int_{r+ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r} \frac{r_0^2\,d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\,dr_0
\end{aligned}$$

Make the following substitution.

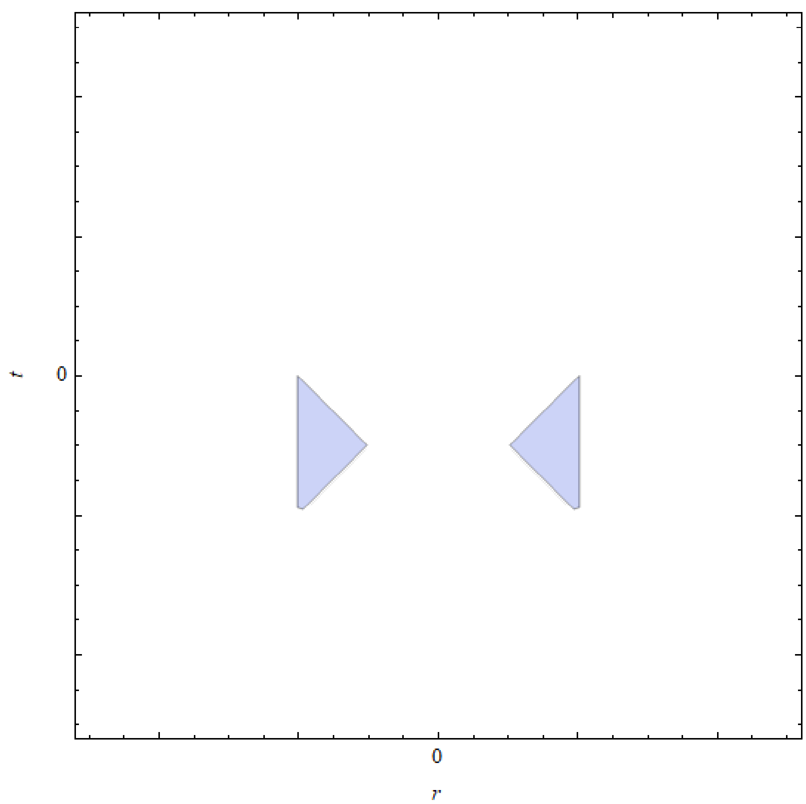
$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$

$$ds_0 = -2r_0\,d\gamma_0 \quad \to \quad -\frac{ds_0}{2} = r_0\,d\gamma_0$$

Therefore,

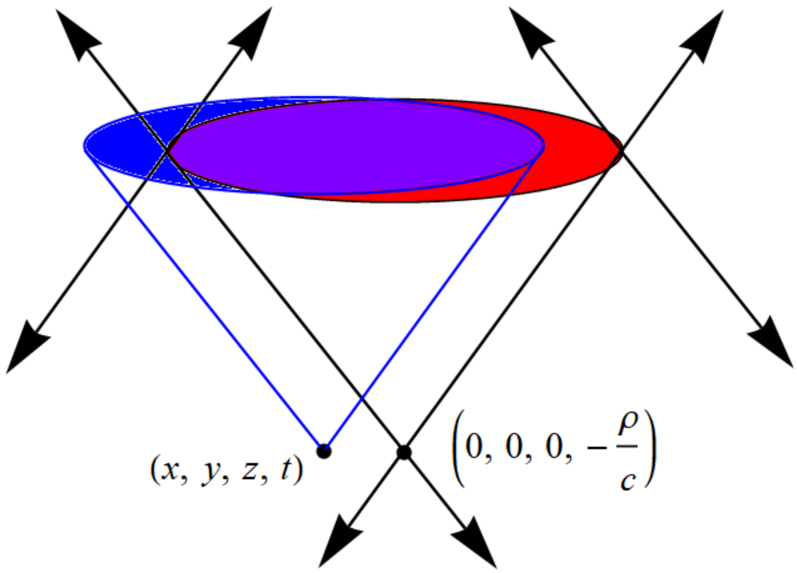
$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0\left(\int_{c^2t^2}^{r_0^2-2r_0r+r^2}\frac{-ds_0}{2\sqrt{s_0}}\right)dr_0 \\
&= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0\left(\sqrt{c^2t^2}-\sqrt{(r_0-r)^2}\right)dr_0 \\
&= \frac{A}{2c^2r}\int_{r+ct}^{\rho} r_0\left(-ct-|r_0-r|\right)dr_0 \\
&= \frac{A}{2c^2r}\left\{\int_{r+ct}^{r} r_0[-ct-(r-r_0)]\,dr_0+\int_{r}^{\rho} r_0[-ct-(r_0-r)]\,dr_0\right\} \\
&= \frac{A}{2c^2r}\left\{\frac{1}{6}[-r^3+c^3t^3-2\rho^3-3ct(\rho^2-r^2)+3r(\rho^2+c^2t^2)]\right\} \\
&= \frac{A}{12c^2r}\left[-r^3+c^3t^3-2\rho^3-3ct(\rho^2-r^2)+3r(\rho^2+c^2t^2)\right].
\end{aligned}$$

This solution is valid for $|\rho+ct|<r<\rho-ct$ and $r<\rho$ and $r+ct>0$ and $r^2<c^2t^2+\rho^2$. There's no need to consider the alternative case of $r^2>c^2t^2+\rho^2$ because there are no points in space-time that satisfy $|\rho-ct|<r<\rho+ct$ and $r<\rho$ and $r-ct>0$ and $r^2>c^2t^2+\rho^2$. Having said that, the preceding condition of validity can be relaxed to $|\rho+ct|<r<\rho-ct$ and $r<\rho$ and $r+ct>0$. A rough sketch of it in the rt-plane is shown below.

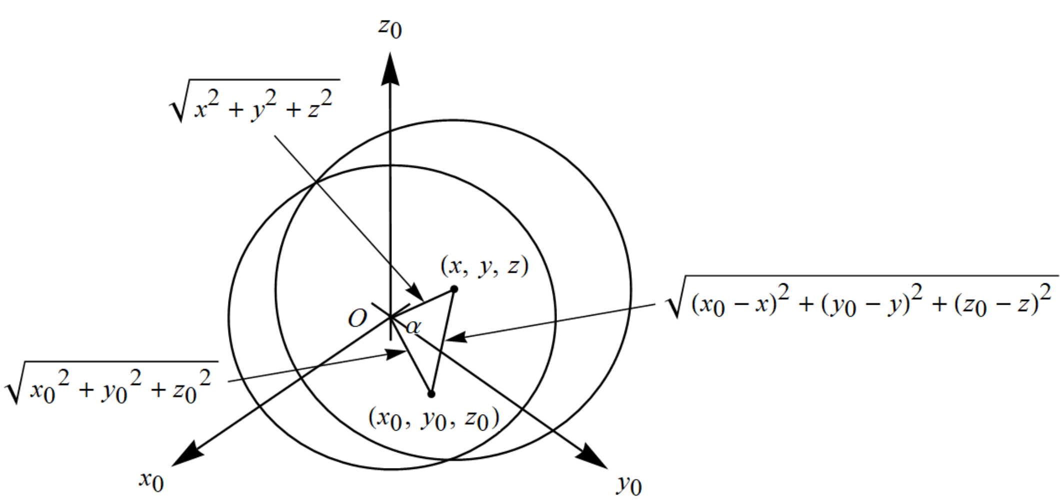


The Magenta Region - Position 4 of 4: $r < \rho$ and $r + ct < 0$

Several different intersections can occur in the magenta region. In this fourth part we will consider a point in space-time (x, y, z, t) inside the cylinder $r = \rho$ and below the cone $r + ct = 0$.



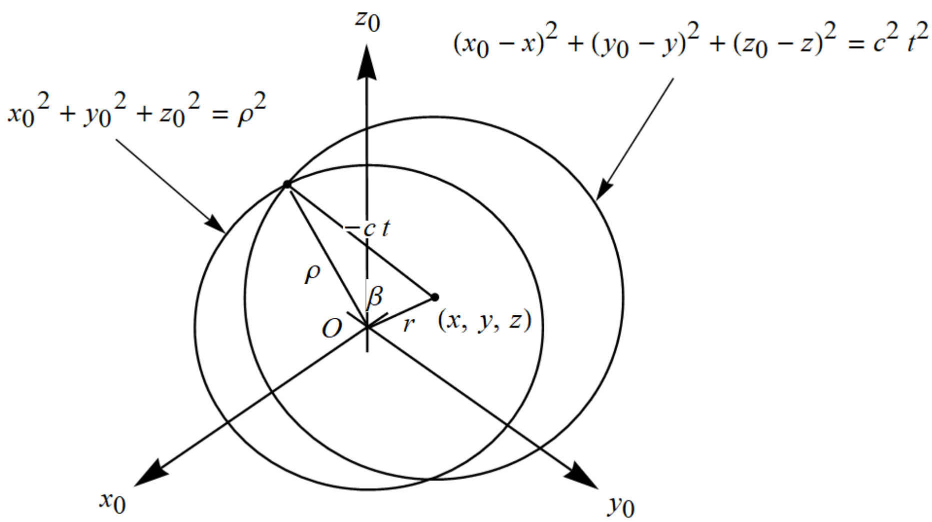
The purple area above is the intersection we have to integrate over. In $x_0y_0z_0$-space, this represents a biconvex lens.



Use the law of cosines to relate the sides of the triangle above with α. Substitute $r = \sqrt{x^2 + y^2 + z^2}$ and $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

$$\begin{aligned}(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 &= (x_0^2 + y_0^2 + z_0^2) + (x^2 + y^2 + z^2) - 2\sqrt{x^2 + y^2 + z^2}\sqrt{x_0^2 + y_0^2 + z_0^2}\cos\alpha \\ &= r_0^2 - 2rr_0\cos\alpha + r^2\end{aligned}$$

Now determine the bounding curves and the angle β at which one curve changes to another.



Use the law of cosines to find β.

$$c^2t^2 = r^2 + \rho^2 - 2\rho r\cos\beta \quad \to \quad \cos\beta = \frac{r^2+\rho^2-c^2t^2}{2\rho r}$$

The bounding curves are

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= \rho^2 & (x_0-x)^2 + (y_0-y)^2 + (z_0-z)^2 &= c^2t^2 \\ r_0^2 &= \rho^2 & r_0^2 - 2rr_0\cos\alpha + r^2 &= c^2t^2 \\ r_0 &= \rho & r_0^2 - (2r\cos\alpha)r_0 - (c^2t^2 - r^2) &= 0 \\ & & r_0 &= \frac{2r\cos\alpha \pm \sqrt{4r^2\cos^2\alpha + 4(c^2t^2-r^2)}}{2} \\ & & r_0 &= r\cos\alpha \pm \sqrt{r^2\cos^2\alpha + (c^2t^2-r^2)} \\ & & r_0 &= r\cos\alpha + \sqrt{r^2\cos^2\alpha + (c^2t^2-r^2)} \end{aligned}$$

The positive sign was chosen because r_0, the distance to (x_0, y_0, z_0), must be positive. As a result of changing to spherical coordinates (r_0, θ_0, ϕ_0) with the polar axis oriented in the direction of (x, y, z), the solution for u becomes

$$\begin{aligned} u(x,y,z,t) &= \frac{A}{4\pi c^2}\iiint\limits_{Q\cap T} \frac{dV_0}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}} \\ &= \frac{A}{4\pi c^2}\left[\int_0^\beta\int_0^{2\pi}\int_0^\rho \frac{r_0^2\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\alpha + r^2}}\right. \\ &\qquad \left. + \int_\beta^\pi\int_0^{2\pi}\int_0^{r\cos\alpha+\sqrt{r^2\cos^2\alpha+(c^2t^2-r^2)}} \frac{r_0^2\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0}{\sqrt{r_0^2-2rr_0\cos\alpha+r^2}}\right]. \end{aligned}$$

Orienting the polar axis as we have makes it so that $\alpha = \phi_0$.

$$u(x,y,z,t) = \frac{A}{4\pi c^2}\left[\int_0^\beta \int_0^{2\pi} \int_0^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \int_\beta^\pi \int_0^{2\pi} \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\theta_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right]$$

Proceed to evaluate the integral in $d\theta_0$.

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{4\pi c^2}\left[\left(\int_0^{2\pi} d\theta_0\right)\int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \left(\int_0^{2\pi} d\theta_0\right)\int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right] \\
&= \frac{A}{2c^2}\left[\int_0^\beta \int_0^\rho \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}} + \int_\beta^\pi \int_0^{r\cos\phi_0 + \sqrt{r^2\cos^2\phi_0 + (c^2t^2 - r^2)}} \frac{r_0^2 \sin\phi_0\, dr_0\, d\phi_0}{\sqrt{r_0^2 - 2rr_0\cos\phi_0 + r^2}}\right]
\end{aligned}$$

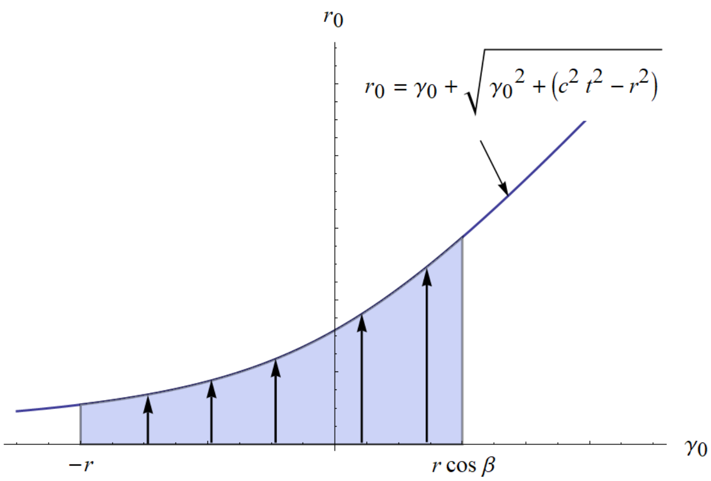
Make the following substitution in both integrals.

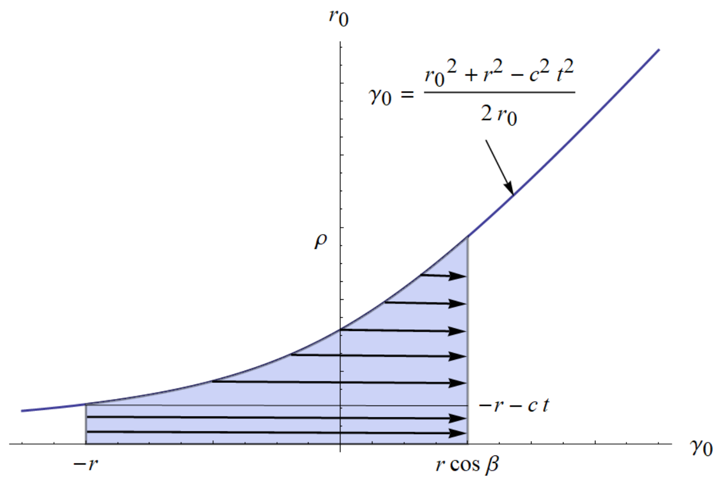
$$\begin{aligned}
\gamma_0 &= r\cos\phi_0 \\
d\gamma_0 &= -r\sin\phi_0\, d\phi_0 \quad \to \quad -\frac{d\gamma_0}{r} = \sin\phi_0\, d\phi_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
u(x,y,z,t) &= \frac{A}{2c^2}\left[\int_r^{r\cos\beta} \int_0^\rho \frac{r_0^2\, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\left(-\frac{d\gamma_0}{r}\right) + \int_{r\cos\beta}^{-r} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2\, dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\left(-\frac{d\gamma_0}{r}\right)\right] \\
&= \frac{A}{2c^2 r}\left[\int_{r\cos\beta}^r \int_0^\rho \frac{r_0^2\, dr_0\, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} + \int_{-r}^{r\cos\beta} \int_0^{\gamma_0 + \sqrt{\gamma_0^2 + (c^2t^2 - r^2)}} \frac{r_0^2\, dr_0\, d\gamma_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right].
\end{aligned}$$

Suppose that $r^2 < c^2t^2 + \rho^2$. Then the current mode of integration in the second integral is shown below on the left.





Integrate over the domain as shown on the right to switch the order of integration.

$$u(x,y,z,t) = \frac{A}{2c^2r}\left(\int_0^\rho \int_{r\cos\beta}^r \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} + \int_0^{-r-ct}\int_{-r}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}} \right.$$
$$\left. + \int_{-r-ct}^{\rho}\int_{\frac{r_0^2+r^2-c^2t^2}{2r_0}}^{r\cos\beta} \frac{r_0^2\,d\gamma_0\,dr_0}{\sqrt{r_0^2 - 2r_0\gamma_0 + r^2}}\right)$$

Make the following substitution in all three integrals.

$$s_0 = r_0^2 - 2r_0\gamma_0 + r^2$$
$$ds_0 = -2r_0\,d\gamma_0 \quad \to \quad -\frac{ds_0}{2} = r_0\,d\gamma_0$$

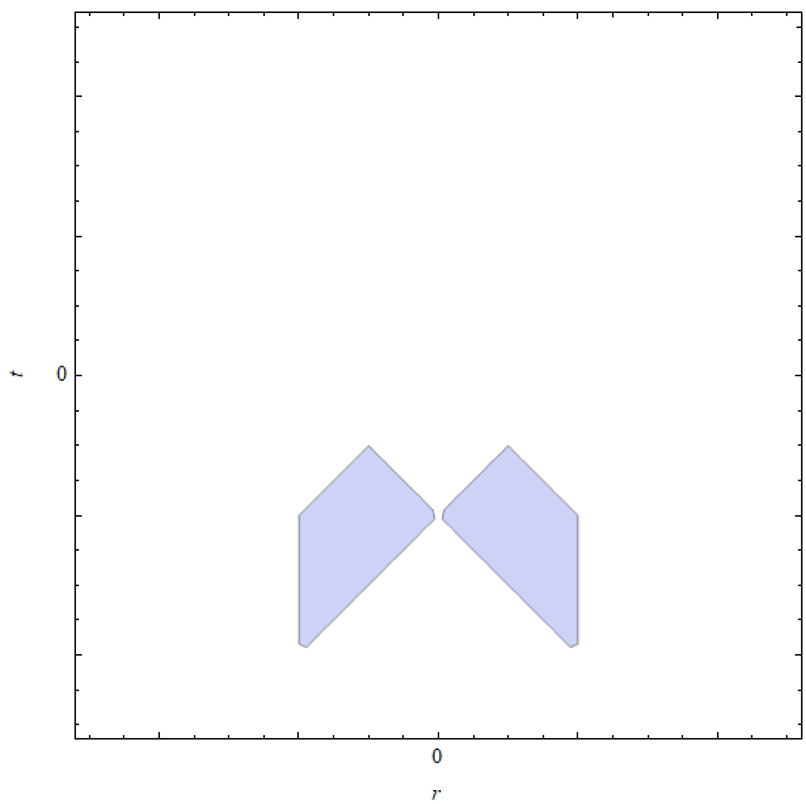
Consequently,

$$u(x,y,z,t) = \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\int_{r_0^2-2r_0r\cos\beta+r^2}^{r_0^2-2r_0r+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0 \right.$$
$$+ \int_0^{-r-ct} r_0\left(\int_{r_0^2+2r_0r+r^2}^{r_0^2-2r_0r\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0$$
$$\left. + \int_{-r-ct}^{\rho} r_0\left(\int_{c^2t^2}^{r_0^2-2r_0r\cos\beta+r^2} \frac{-ds_0}{2\sqrt{s_0}}\right)dr_0\right]$$
$$= \frac{A}{2c^2r}\left[\int_0^\rho r_0\left(\sqrt{r_0^2 - 2r_0r\cos\beta + r^2} - |r_0 - r|\right)dr_0 \right.$$
$$+ \int_0^{-r-ct} r_0\left(r_0 + r - \sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\right)dr_0$$
$$\left. + \int_{-r-ct}^{\rho} r_0\left(-ct - \sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\right)dr_0\right].$$

Therefore,

$$
\begin{aligned}
u(x,y,z,t) = \frac{A}{2c^2r}\bigg[&\int_0^\rho r_0\sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\,dr_0 - \int_0^\rho r_0|r_0 - r|\,dr_0 \\
&+ \int_0^{-r-ct} r_0(r_0+r)\,dr_0 - \int_0^{-r-ct} r_0\sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\,dr_0 \\
&- \int_{-r-ct}^\rho r_0ct\,dr_0 - \int_{-r-ct}^\rho r_0\sqrt{r_0^2 - 2r_0r\cos\beta + r^2}\,dr_0\bigg] \\
= \frac{A}{2c^2r}&\left[-\int_0^\rho r_0|r_0 - r|\,dr_0 + \int_0^{-r-ct} r_0(r_0+r)\,dr_0 - \int_{-r-ct}^\rho r_0ct\,dr_0\right] \\
= \frac{A}{2c^2r}&\left[-\int_0^r r_0(r - r_0)\,dr_0 - \int_r^\rho r_0(r_0 - r)\,dr_0 + \int_0^{-r-ct} r_0(r_0+r)\,dr_0 - \int_{-r-ct}^\rho r_0ct\,dr_0\right] \\
= \frac{A}{2c^2r}&\left\{\frac{1}{6}[-r^3 + c^3t^3 - 2\rho^3 - 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)]\right\} \\
= \frac{A}{12c^2r}&\left[-r^3 + c^3t^3 - 2\rho^3 - 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right].
\end{aligned}
$$

This solution is valid for $|\rho + ct| < r < \rho - ct$ and $r < \rho$ and $r + ct < 0$ and $r^2 < c^2t^2 + \rho^2$. There's no need to consider the alternative case of $r^2 > c^2t^2 + \rho^2$ because there are no points in space-time that satisfy $|\rho + ct| < r < \rho - ct$ and $r < \rho$ and $r + ct < 0$ and $r^2 > c^2t^2 + \rho^2$. Having said that, the preceding condition of validity can be relaxed to $|\rho + ct| < r < \rho - ct$ and $r < \rho$ and $r + ct < 0$. A rough sketch of it in the rt-plane is shown below.



The results obtained so far are summarized here in the same order.

$$
u(x,y,z,t) = \begin{cases}
0 & \text{if } r > \rho + c|t| \\
\frac{1}{2}At^2 & \text{if } r + ct < \rho \\
\frac{1}{2}At^2 & \text{if } r - ct < \rho \\
\frac{A}{6c^2}(3\rho^2 - r^2) & \text{if } ct > r + \rho \text{ and } r < \rho \\
\frac{A\rho^3}{3c^2 r} & \text{if } ct > r + \rho \text{ and } r > \rho \\
\frac{A}{6c^2}(3\rho^2 - r^2) & \text{if } -ct > r + \rho \text{ and } r < \rho \\
\frac{A\rho^3}{3c^2 r} & \text{if } -ct > r + \rho \text{ and } r > \rho \\
\frac{A}{12c^2 r}\left[r^3 - c^3t^3 + 2\rho^3 - 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r > \rho \text{ and } r - ct > 0 \\
 & \qquad \text{and } r^2 > c^2t^2 + \rho^2 \\
\frac{A}{48\rho^3 c^2 r}\left\{3(r^2 + \rho^2 - c^2t^2)[4\rho^2 r^2 - (r^2 + \rho^2 - c^2t^2)^2]\tanh^{-1}\frac{c^2t^2 + \rho^2 - r^2}{2ct\rho}\right. & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r > \rho \\
\qquad \left. + 2\rho\left[3ct(r^2 - c^2t^2)^2 + 2\rho^2(r - ct)^3 - 3\rho^4(2r - ct) + 4\rho^5\right]\right\} & \qquad \text{and } r - ct > 0 \text{ and } r^2 < c^2t^2 + \rho^2 \\
\frac{A}{12c^2 r}\left[r^3 - c^3t^3 + 2\rho^3 - 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r > \rho \text{ and } r - ct < 0 \\
\frac{A}{12c^2 r}\left[-r^3 - c^3t^3 - 2\rho^3 + 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r < \rho \text{ and } r - ct > 0 \\
\frac{A}{12c^2 r}\left[-r^3 - c^3t^3 - 2\rho^3 + 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r < \rho \text{ and } r - ct < 0 \\
\frac{A}{12c^2 r}\left[r^3 + c^3t^3 + 2\rho^3 + 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r > \rho \text{ and } r + ct > 0 \\
 & \qquad \text{and } r^2 > c^2t^2 + \rho^2 \\
\frac{A}{48\rho^3 c^2 r}\left\{3(r^2 + \rho^2 - c^2t^2)[4\rho^2 r^2 - (r^2 + \rho^2 - c^2t^2)^2]\tanh^{-1}\frac{c^2t^2 + \rho^2 - r^2}{2ct\rho}\right. & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r > \rho \\
\qquad \left. + 2\rho\left[-3ct(r^2 - c^2t^2)^2 + 2\rho^2(r + ct)^3 - 3\rho^4(2r + ct) + 4\rho^5\right]\right\} & \qquad \text{and } r + ct > 0 \text{ and } r^2 < c^2t^2 + \rho^2 \\
\frac{A}{12c^2 r}\left[r^3 + c^3t^3 + 2\rho^3 + 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r > \rho \text{ and } r + ct < 0 \\
\frac{A}{12c^2 r}\left[-r^3 + c^3t^3 - 2\rho^3 - 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r < \rho \text{ and } r + ct > 0 \\
\frac{A}{12c^2 r}\left[-r^3 + c^3t^3 - 2\rho^3 - 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r < \rho \text{ and } r + ct < 0
\end{cases}
$$

Notice that several of these solutions are identical. Combine their conditions of validity by using absolute value signs where necessary.

$$u(x,y,z,t) = \begin{cases} 0 & \text{if } r > \rho + c|t| \\ \dfrac{1}{2}At^2 & \text{if } r + c|t| < \rho \\ \dfrac{A}{6c^2}(3\rho^2 - r^2) & \text{if } c|t| > r + \rho \text{ and } r < \rho \\ \dfrac{A\rho^3}{3c^2 r} & \text{if } c|t| > r + \rho \text{ and } r > \rho \\ \dfrac{A}{12c^2 r}\left[r^3 - c^3t^3 + 2\rho^3 - 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r > \rho \text{ and } [r - ct < 0 \\ & \qquad \text{or } (r - ct > 0 \text{ and } r^2 > c^2t^2 + \rho^2)] \\ \dfrac{A}{12c^2 r}\left[-r^3 - c^3t^3 - 2\rho^3 + 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r < \rho \\ \dfrac{A}{12c^2 r}\left[r^3 + c^3t^3 + 2\rho^3 + 3ct(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r > \rho \text{ and } [r + ct < 0 \\ & \qquad \text{or } (r + ct > 0 \text{ and } r^2 > c^2t^2 + \rho^2)] \\ \dfrac{A}{12c^2 r}\left[-r^3 + c^3t^3 - 2\rho^3 - 3ct(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r < \rho \\ \dfrac{A}{48\rho^3c^2r}\left\{3(r^2 + \rho^2 - c^2t^2)[4\rho^2r^2 - (r^2 + \rho^2 - c^2t^2)^2]\tanh^{-1}\dfrac{c^2t^2 + \rho^2 - r^2}{2ct\rho}\right. & \text{if } |\rho - ct| < r < \rho + ct \text{ and } r > \rho \\ \qquad \left. + 2\rho\left[3ct(r^2 - c^2t^2)^2 + 2\rho^2(r - ct)^3 - 3\rho^4(2r - ct) + 4\rho^5\right]\right\} & \qquad \text{and } r - ct > 0 \text{ and } r^2 < c^2t^2 + \rho^2 \\ \dfrac{A}{48\rho^3c^2r}\left\{3(r^2 + \rho^2 - c^2t^2)[4\rho^2r^2 - (r^2 + \rho^2 - c^2t^2)^2]\tanh^{-1}\dfrac{c^2t^2 + \rho^2 - r^2}{2ct\rho}\right. & \text{if } |\rho + ct| < r < \rho - ct \text{ and } r > \rho \\ \qquad \left. + 2\rho\left[-3ct(r^2 - c^2t^2)^2 + 2\rho^2(r + ct)^3 - 3\rho^4(2r + ct) + 4\rho^5\right]\right\} & \qquad \text{and } r + ct > 0 \text{ and } r^2 < c^2t^2 + \rho^2 \end{cases}$$

Fuse more of the solutions.

$$u(x,y,z,t) = \begin{cases} 0 & \text{if } r > \rho + c|t| \\ \dfrac{1}{2}At^2 & \text{if } r + c|t| < \rho \\ \dfrac{A}{6c^2}(3\rho^2 - r^2) & \text{if } c|t| > r + \rho \text{ and } r < \rho \\ \dfrac{A\rho^3}{3c^2 r} & \text{if } c|t| > r + \rho \text{ and } r > \rho \\ \dfrac{A}{12c^2 r}\left[r^3 - c^3|t|^3 + 2\rho^3 - 3c|t|(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho - c|t|| < r < \rho + c|t| \text{ and } r > \rho \text{ and} \\ & [r - c|t| < 0 \text{ or } (r - c|t| > 0 \text{ and } r^2 > c^2t^2 + \rho^2)] \\ \dfrac{A}{12c^2 r}\left[-r^3 - c^3|t|^3 - 2\rho^3 + 3c|t|(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho - c|t|| < r < \rho + c|t| \text{ and } r < \rho \\ \dfrac{A}{48\rho^3 c^2 r}\Big\{3(r^2 + \rho^2 - c^2t^2)[4\rho^2 r^2 - (r^2 + \rho^2 - c^2t^2)^2]\tanh^{-1}\dfrac{c^2t^2 + \rho^2 - r^2}{2ct\rho} & \text{if } |\rho - c|t|| < r < \rho + c|t| \text{ and } r > \rho \\ \qquad + 2\rho\left[3c|t|(r^2 - c^2t^2)^2 + 2\rho^2(r - c|t|)^3 - 3\rho^4(2r - c|t|) + 4\rho^5\right]\Big\} & \quad \text{and } r - c|t| > 0 \text{ and } r^2 < c^2t^2 + \rho^2 \end{cases}$$

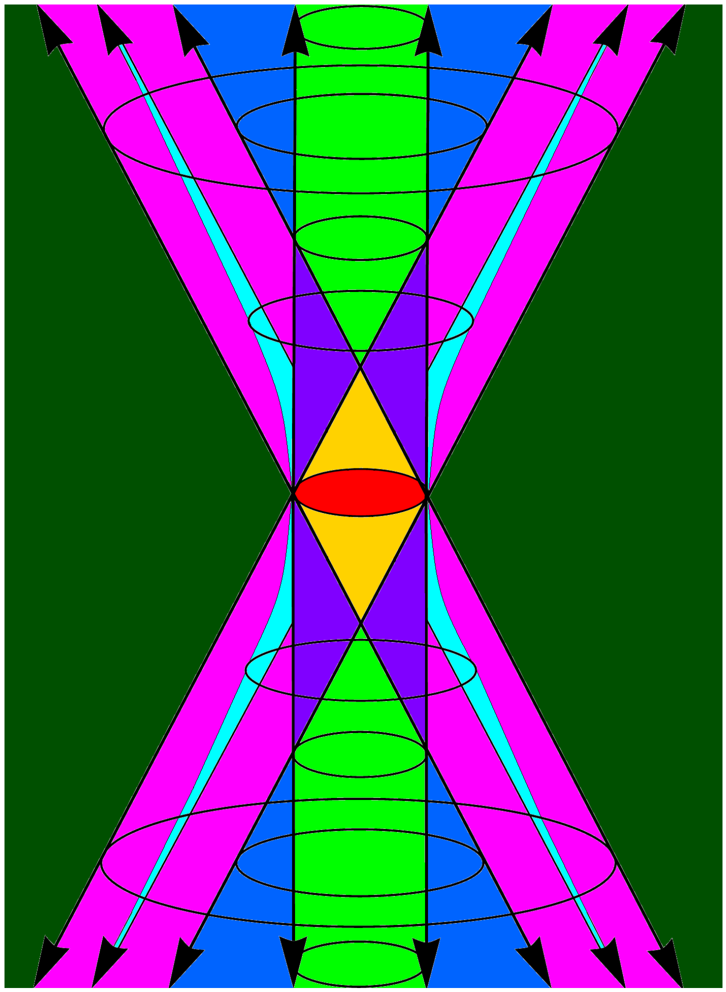
Now we will check to see if the solutions are continuous across the boundaries. If we plug in $r = \rho + c|t|$ into the fifth entry, then we get 0, the first entry. Substituting $r = c|t| - \rho$ into the fifth entry yields the fourth solution. Plugging in $r = \rho$ into the third and fourth entries results in the same solution as well. Plugging in $r = \rho - c|t|$ into the sixth entry gives $(1/2)At^2$, the second solution. In addition, plugging in $r = c|t| - \rho$ into the sixth entry yields the third solution. Substituting $r = c|t| - \rho$ into the fifth entry results in the fourth entry. Substituting $r^2 = c^2t^2 + \rho^2$ in the last entry results in the fifth solution.

In conclusion,

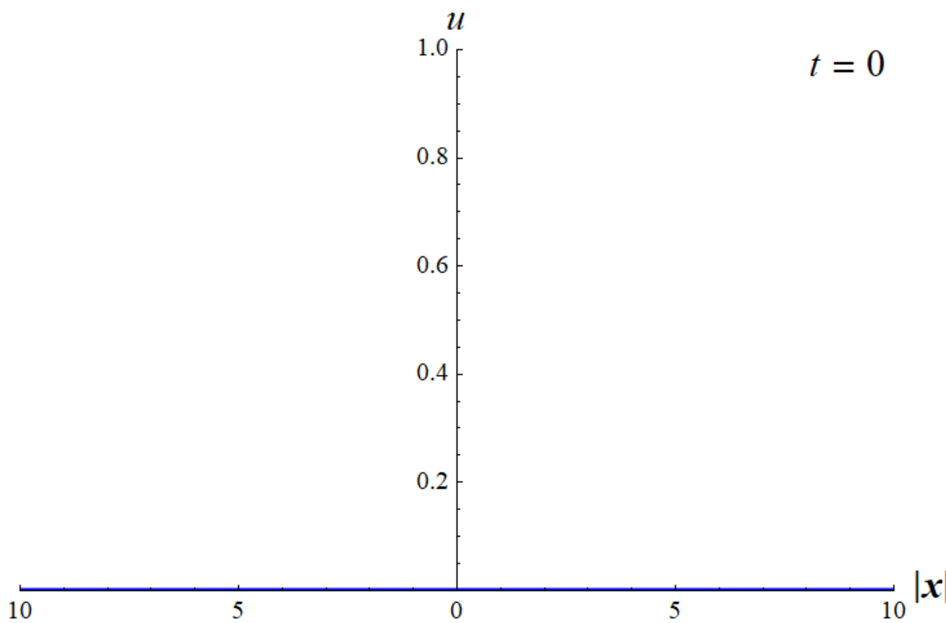
$$u(x,y,z,t) = \begin{cases} 0 & \text{if } r \geq \rho + c|t| \\ \dfrac{1}{2}At^2 & \text{if } r + c|t| \leq \rho \\ \dfrac{A}{6c^2}(3\rho^2 - r^2) & \text{if } c|t| \geq r + \rho \text{ and } r \leq \rho \\ \dfrac{A\rho^3}{3c^2 r} & \text{if } c|t| \geq r + \rho \text{ and } r \geq \rho \\ \dfrac{A}{12c^2 r}\left[r^3 - c^3|t|^3 + 2\rho^3 - 3c|t|(r^2 - \rho^2) + 3r(c^2t^2 - \rho^2)\right] & \text{if } |\rho - c|t|| \leq r \leq \rho + c|t| \text{ and } r > \rho \text{ and} \\ & \quad [r - c|t| < 0 \text{ or } (r - c|t| > 0 \text{ and } r^2 \geq c^2t^2 + \rho^2)] \\ \dfrac{A}{12c^2 r}\left[-r^3 - c^3|t|^3 - 2\rho^3 + 3c|t|(\rho^2 - r^2) + 3r(\rho^2 + c^2t^2)\right] & \text{if } |\rho - c|t|| \leq r < \rho + c|t| \text{ and } r < \rho \\ \dfrac{A}{48\rho^3 c^2 r}\left\{3(r^2 + \rho^2 - c^2t^2)[4\rho^2 r^2 - (r^2 + \rho^2 - c^2t^2)^2] \tanh^{-1}\dfrac{c^2t^2 + \rho^2 - r^2}{2ct\rho}\right. & \text{if } |\rho - c|t|| < r < \rho + c|t| \text{ and } r > \rho \\ \qquad \left. + 2\rho\left[3c|t|(r^2 - c^2t^2)^2 + 2\rho^2(r - c|t|)^3 - 3\rho^4(2r - c|t|) + 4\rho^5\right]\right\} & \quad \text{and } r - c|t| > 0 \text{ and } r^2 \leq c^2t^2 + \rho^2 \end{cases}$$

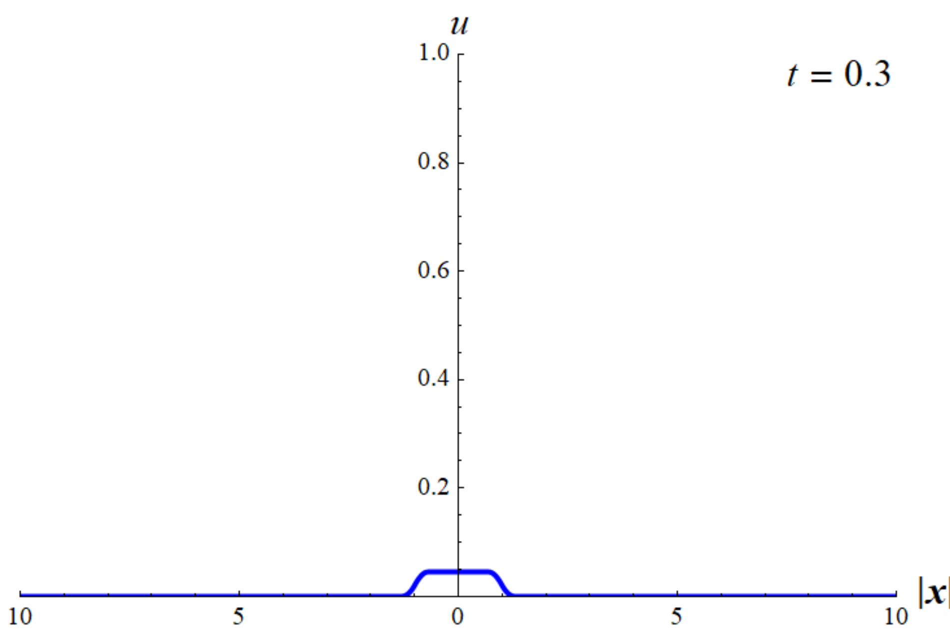
where $r = \sqrt{x^2 + y^2 + z^2}$.

The solutions are newly labelled in each part of space-time by color. The origin of the $x_0y_0z_0t_0$-coordinate system is at the center of the red hyperdisk, which has radius ρ.



Note that the bounding surface of the cyan region here is a hyperboloid of one sheet; it is asymptotic to the cone $x_0^2 + y_0^2 + z_0^2 = c^2t_0^2$. In addition, the vertices of the golden cones are $(0, 0, 0, \pm\rho/c)$. The royal blue regions begin at $t = \pm 2\rho/c$. A sixteen-frame movie will now be produced for the special case of $c = 1$, $\rho = 1$, and $A = 1$ to illustrate the behavior of the solution.


u
1.0
0.8
0.6
0.4
0.2
t = 0
10
5
0
5
10
|x|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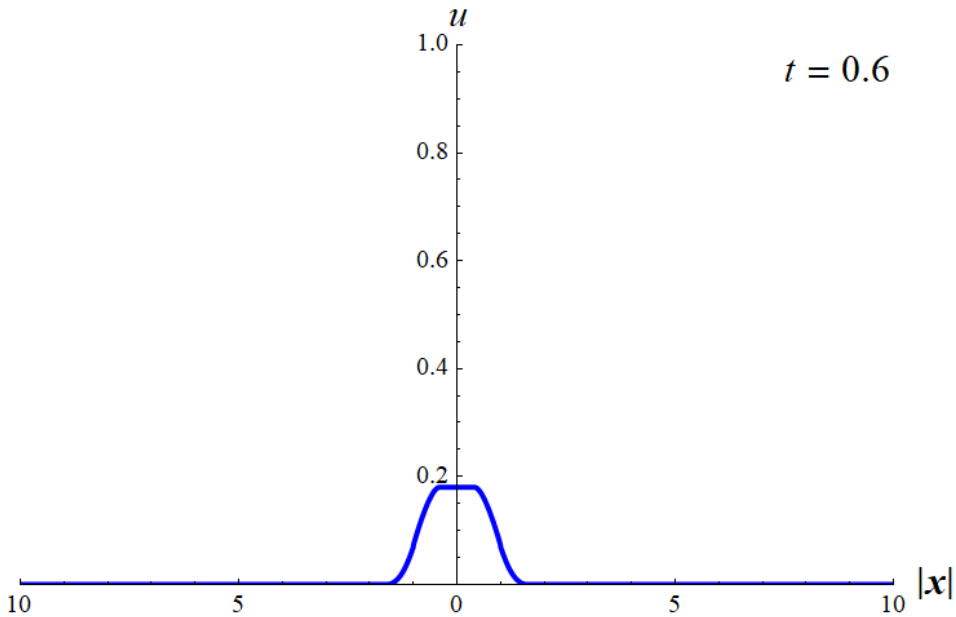
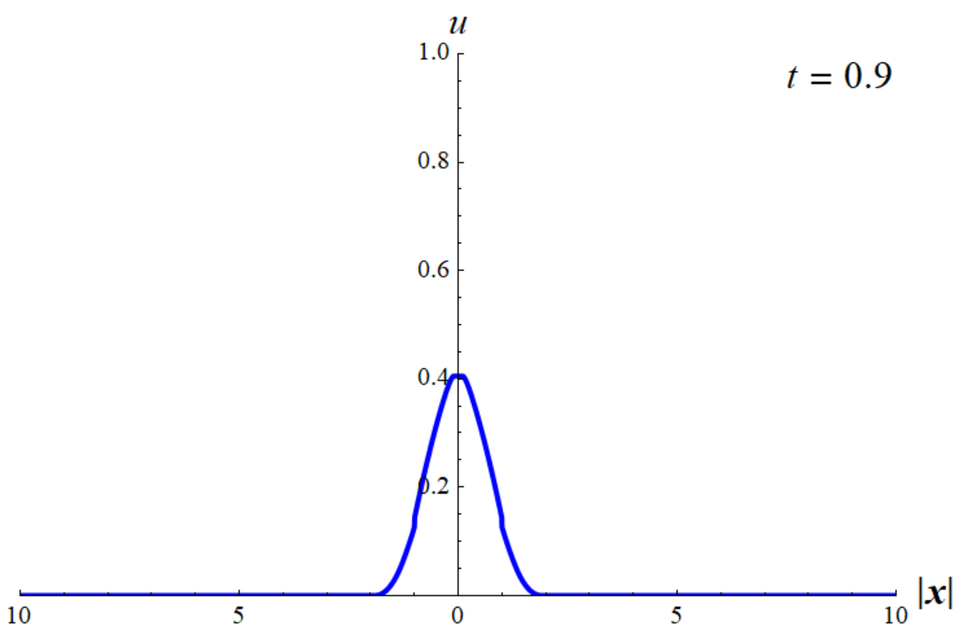
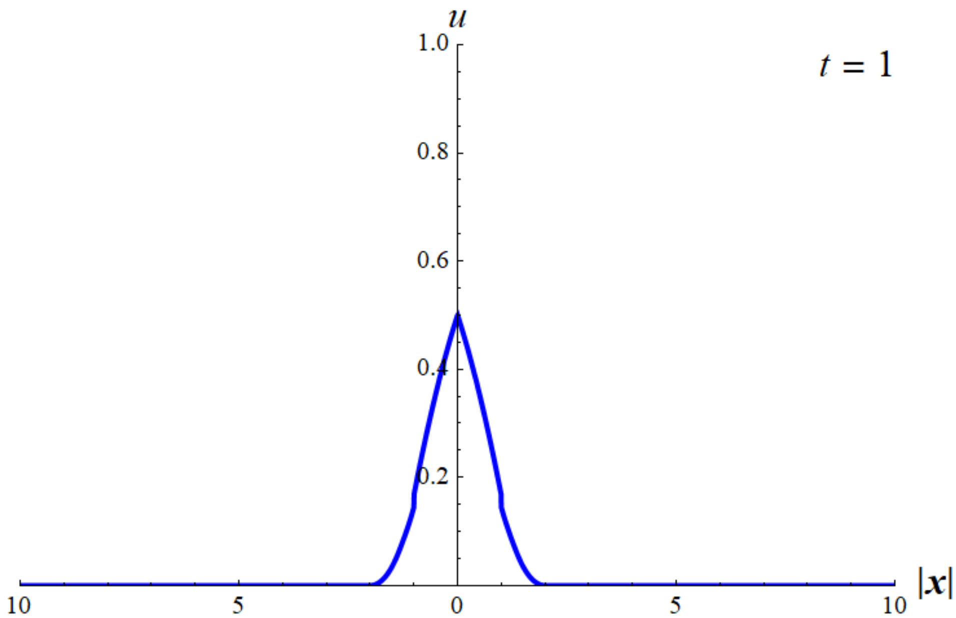
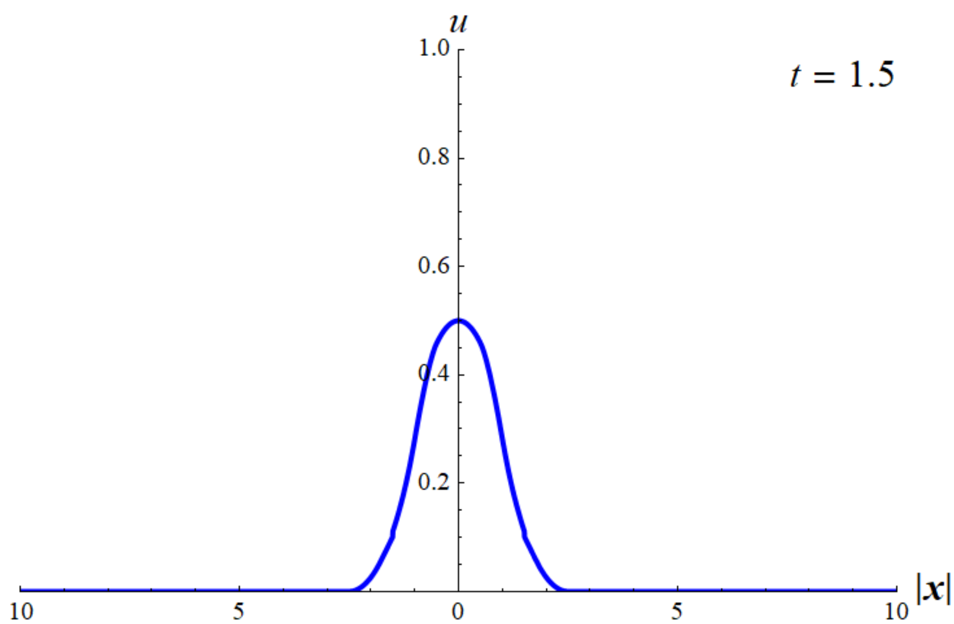
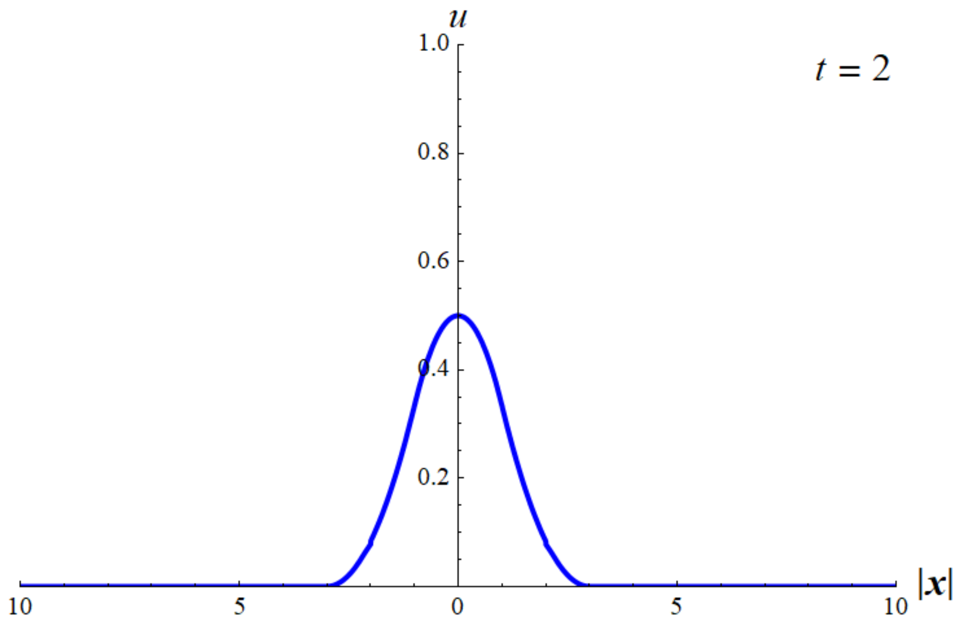
0.8
0.6
0.4
0.2
t = 0.3
10
5
0
5
10
|x|


u
1.0
0.8
0.6
0.4
0.2
t = 0.6
10
5
0
5
10
|x|

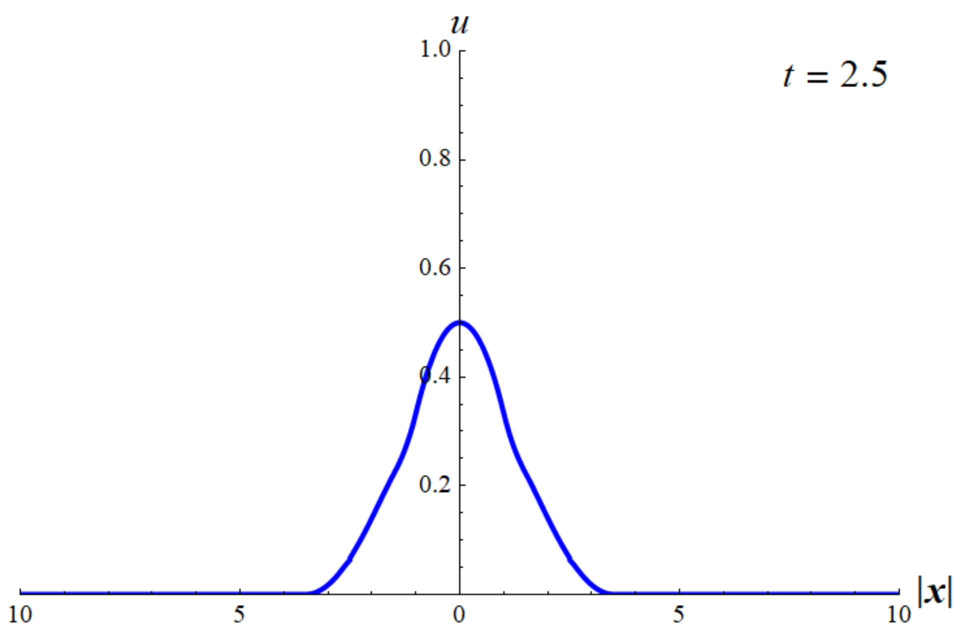

u
1.0
0.8
0.6
0.4
0.2
t = 0.9
10
5
0
5
10
|x|


u
1.0
0.8
0.6
0.4
0.2
t = 1
|x|
10
5
0
5
10

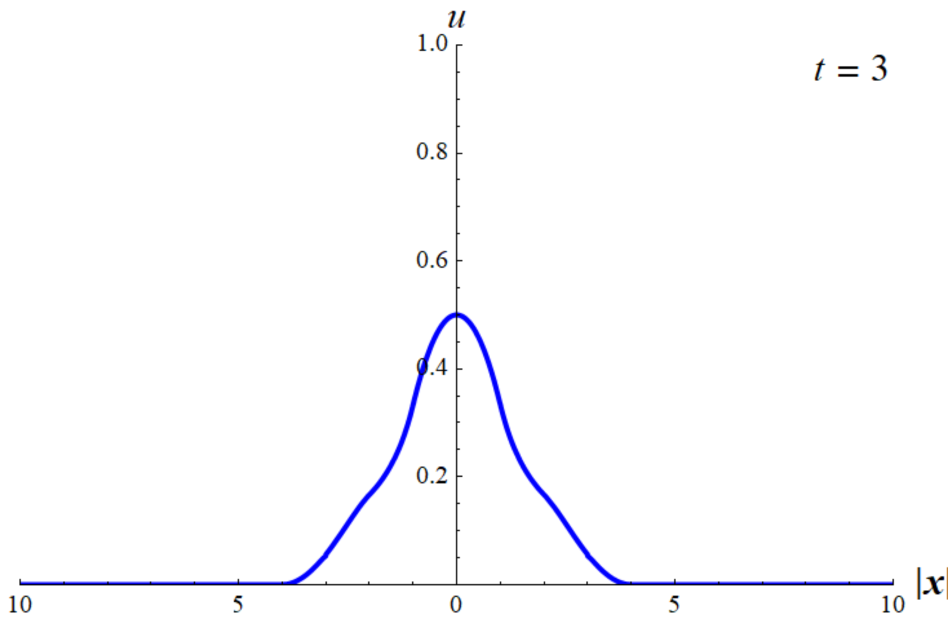

u
1.0
0.8
0.6
0.4
0.2
t = 1.5
|x|
10
5
0
5
10

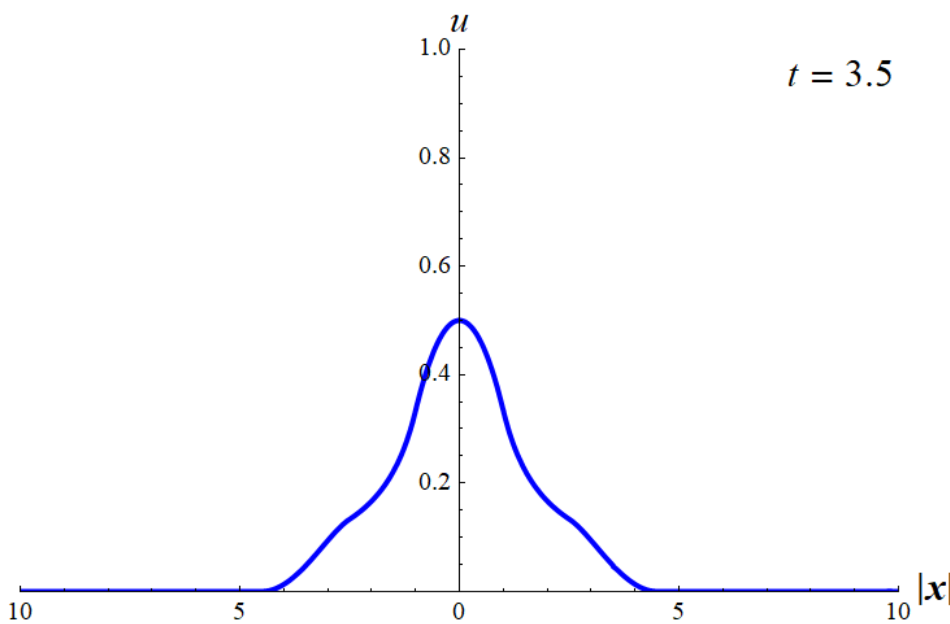


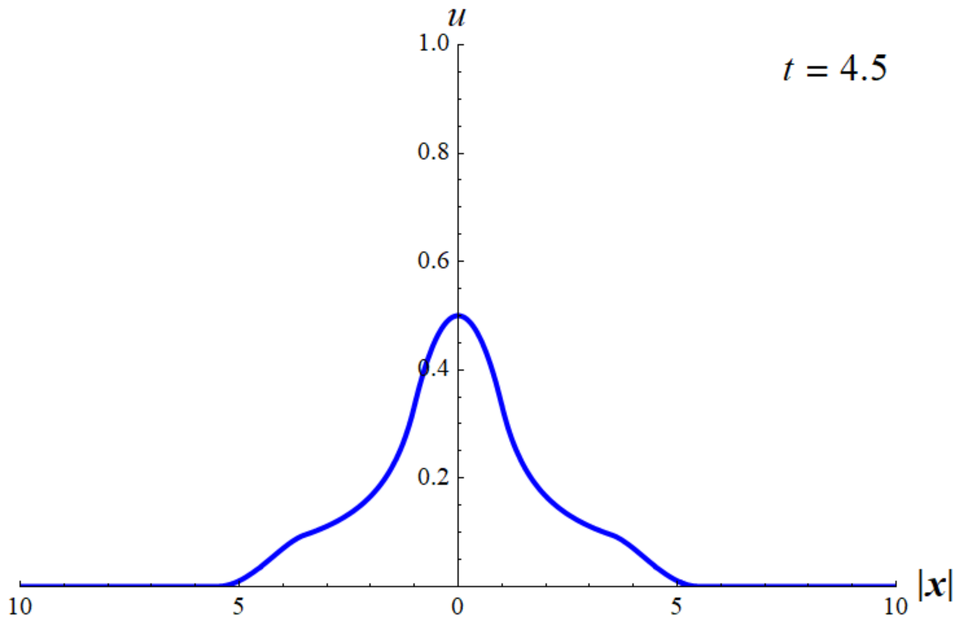
u
1.0
0.8
0.6
0.4
0.2
t = 2
10
5
0
5
10
|x|

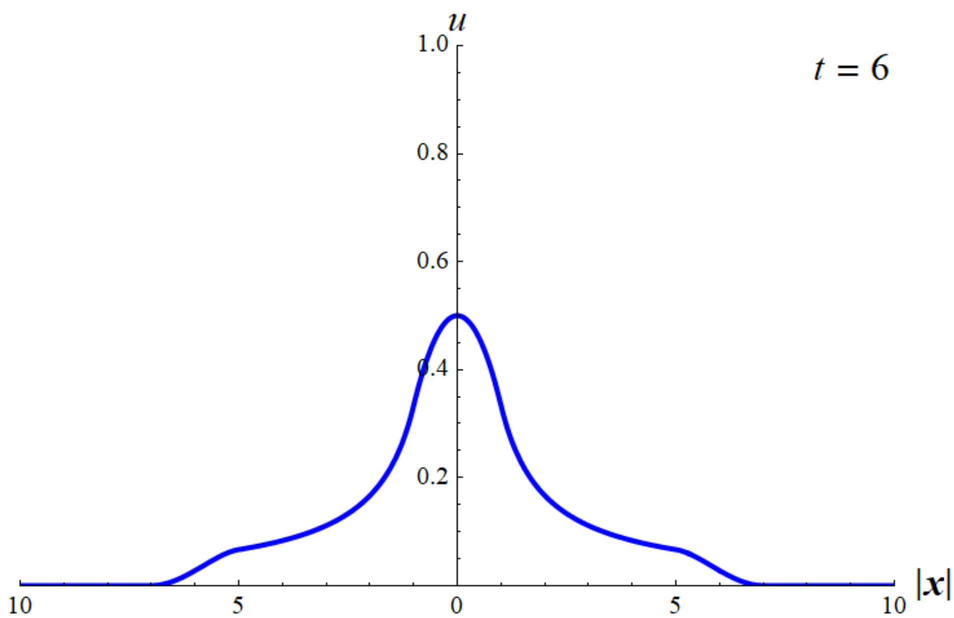


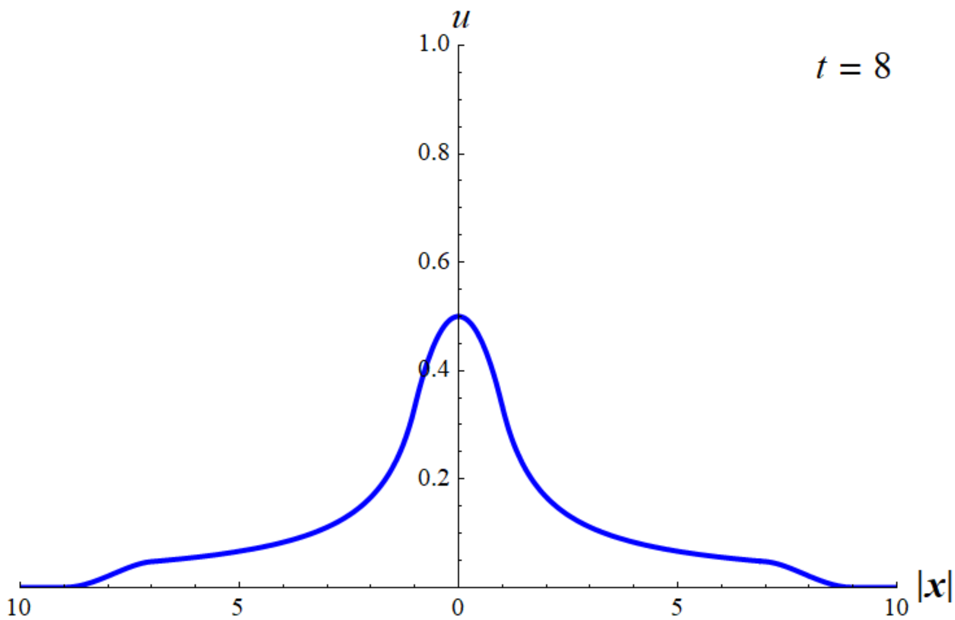
u
1.0
0.8
0.6
0.4
0.2
t = 2.5
10
5
0
5
10
|x|

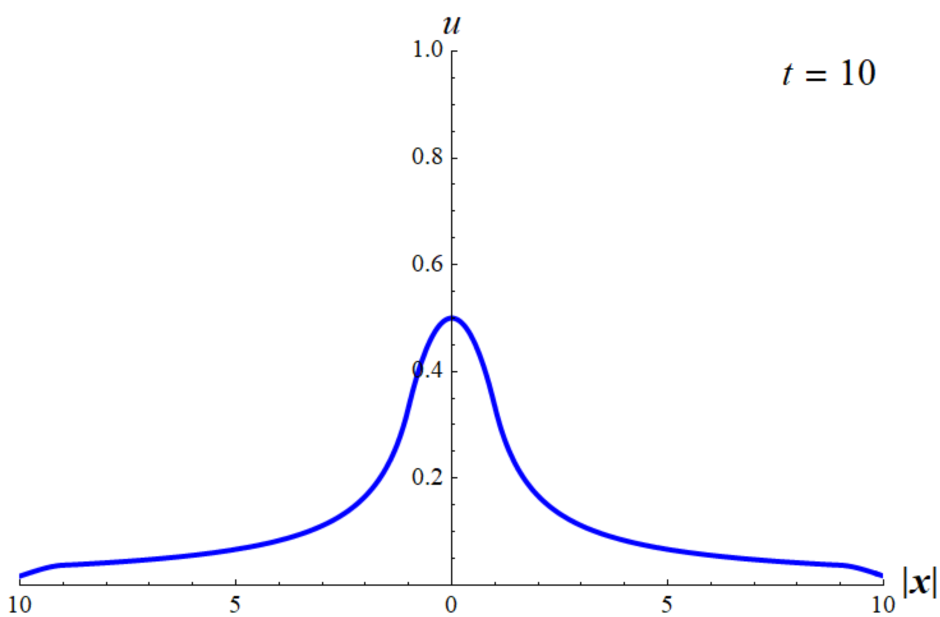

u
1.0
0.8
0.6
0.4
0.2
t = 3
10
5
0
5
10
|x|

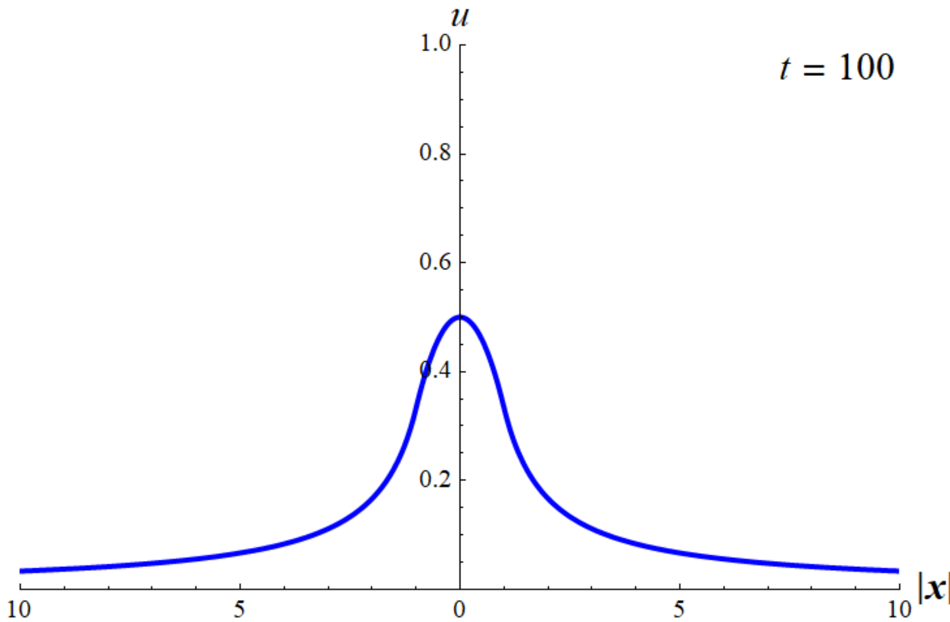

u
1.0
0.8
0.6
0.4
0.2
t = 3.5
10
5
0
5
10
|x|


u
1.0
0.8
0.6
0.4
0.2
$t = 4.5$
10
5
0
5
10
|x|

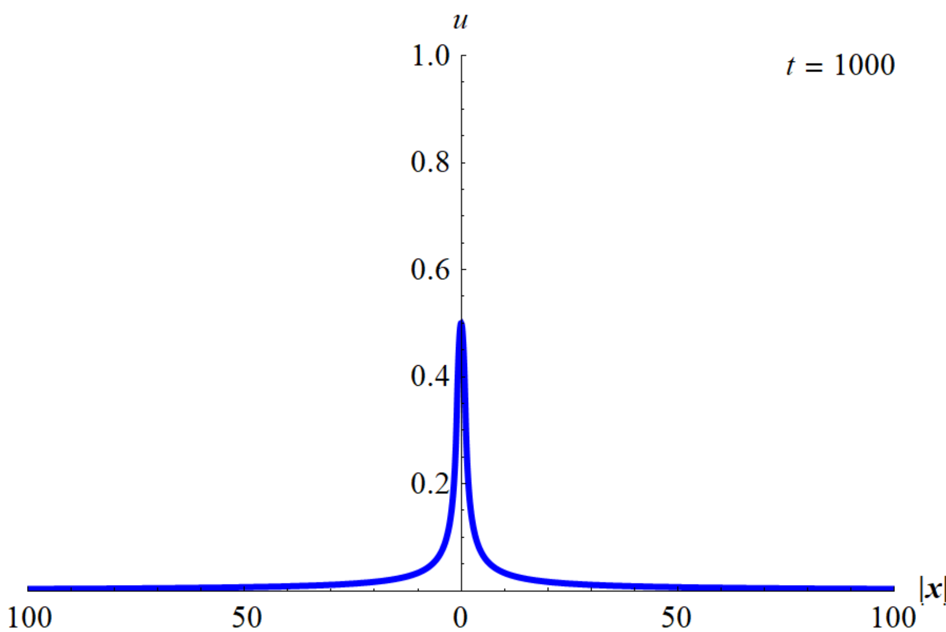

u
1.0
0.8
0.6
0.4
0.2
$t = 6$
10
5
0
5
10
|x|


u
1.0
0.8
0.6
0.4
0.2
t = 8
10
5
0
5
10
|x|


u
1.0
0.8
0.6
0.4
0.2
t = 10
10
5
0
5
10
|x|



u
1.0
0.8
0.6
0.4
0.2
t = 100
|x|
10
5
0
5
10



u
1.0
0.8
0.6
0.4
0.2
t = 1000
|x|
100
50
0
50
100

The Final Integral

Here we will evaluate the last integral that came as a result of assuming $r^2 < c^2t^2 + \rho^2$ in the first part of the cyan and magenta regions. Start by completing the square under the square root.

$$\begin{aligned}
I &= \int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2 - 2r_0 r\cos\beta + r^2}\, dr_0 \\
&= \int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{r_0^2 - 2r_0 r\cos\beta + r^2\cos^2\beta + r^2 - r^2\cos^2\beta}\, dr_0 \\
&= \int_{\frac{r^2-c^2t^2}{\rho}}^{\rho} r_0\sqrt{(r_0 - r\cos\beta)^2 + r^2\sin^2\beta}\, dr_0
\end{aligned}$$

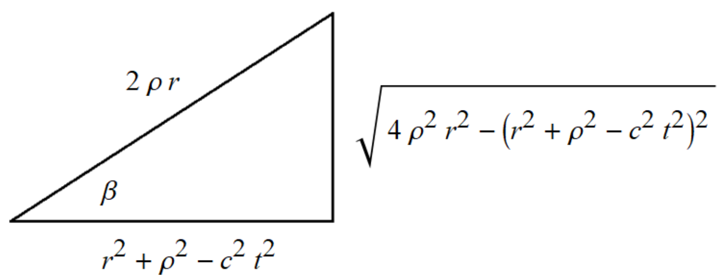
Make a trigonometric substitution to simplify the integrand.

$$\begin{aligned}
r_0 - r\cos\beta &= r\sin\beta\tan\psi_0 & &\rightarrow & &(r_0 - r\cos\beta)^2 + r^2\sin^2\beta = r^2\sin^2\beta\sec^2\psi_0 \\
r_0 &= r\cos\beta + r\sin\beta\tan\psi_0 & &\rightarrow & &r_0 = r\cos\beta(1+\tan\beta\tan\psi_0) \\
dr_0 &= r\sin\beta\sec^2\psi_0\, d\psi_0
\end{aligned}$$

Recall that

$$\cos\beta = \frac{r^2+\rho^2-c^2t^2}{2\rho r}.$$

This implies the following right triangle.



As a result,

$$\tan\psi_0 = \frac{r_0 - r\cos\beta}{r\sin\beta} = \frac{r_0 - \frac{r^2+\rho^2-c^2t^2}{2\rho}}{\frac{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}}{2\rho}} = \frac{2\rho r_0 + c^2t^2 - \rho^2 - r^2}{\sqrt{4\rho^2r^2 - (r^2+\rho^2-c^2t^2)^2}}.$$

$$\begin{aligned}
&\text{If } r_0 = \rho & &\text{then} & \tan\psi_0 &= \frac{c^2t^2+\rho^2-r^2}{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}}. \\
&\text{If } r_0 = \frac{r^2-c^2t^2}{\rho} & &\text{then} & \tan\psi_0 &= -\frac{c^2t^2+\rho^2-r^2}{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}}.
\end{aligned}$$

The integral becomes

$$I = \int_{-\psi}^{\psi} r\cos\beta(1+\tan\beta\tan\psi_0)\sqrt{r^2\sin^2\beta\sec^2\psi_0}(r\sin\beta\sec^2\psi_0\, d\psi_0),$$

where ψ is an angle defined as

$$\psi = \tan^{-1} \frac{c^2t^2 + \rho^2 - r^2}{\sqrt{4\rho^2 r^2 - (r^2 + \rho^2 - c^2t^2)^2}}.$$

Simplify the integrand, bring the constants in front, and use a table of integrals to find the antiderivative.

$$\begin{aligned}
I &= \int_{-\psi}^{\psi} r\cos\beta(1 + \tan\beta\tan\psi_0)(r\sin\beta\sec\psi_0)(r\sin\beta\sec^2\psi_0\, d\psi_0) \\
&= \int_{-\psi}^{\psi} r^3\sin^2\beta\cos\beta(1 + \tan\beta\tan\psi_0)\sec^3\psi_0\, d\psi_0 \\
&= r^3\sin^2\beta\cos\beta \int_{-\psi}^{\psi} (1 + \tan\beta\tan\psi_0)\sec^3\psi_0\, d\psi_0 \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\left(3\ln\frac{\cos\frac{\psi_0}{2} + \sin\frac{\psi_0}{2}}{\cos\frac{\psi_0}{2} - \sin\frac{\psi_0}{2}} + 2\tan\beta\sec^3\psi_0 + 3\sec\psi_0\tan\psi_0\right)\Bigg|_{-\psi}^{\psi} \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\Bigg[3\ln\frac{\cos\frac{\psi}{2} + \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} - \sin\frac{\psi}{2}} - 3\ln\frac{\cos\frac{-\psi}{2} + \sin\frac{-\psi}{2}}{\cos\frac{-\psi}{2} - \sin\frac{-\psi}{2}} \\
&\qquad + 2\tan\beta\sec^3\psi - 2\tan\beta[\sec(-\psi)]^3 + 3\sec\psi\tan\psi - 3\sec(-\psi)\tan(-\psi)\Bigg] \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\Bigg(3\ln\frac{\cos\frac{\psi}{2} + \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} - \sin\frac{\psi}{2}} - 3\ln\frac{\cos\frac{\psi}{2} - \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} + \sin\frac{\psi}{2}} \\
&\qquad + 2\tan\beta\sec^3\psi - 2\tan\beta\sec^3\psi + 3\sec\psi\tan\psi + 3\sec\psi\tan\psi\Bigg) \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\left(3\ln\frac{\cos\frac{\psi}{2} + \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} - \sin\frac{\psi}{2}} - 3\ln\frac{\cos\frac{\psi}{2} - \sin\frac{\psi}{2}}{\cos\frac{\psi}{2} + \sin\frac{\psi}{2}} + 6\sec\psi\tan\psi\right) \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\left[3\ln\left(\frac{\sin\frac{\psi}{2}}{\sin\frac{\psi}{2}}\cdot\frac{\cot\frac{\psi}{2} + 1}{\cot\frac{\psi}{2} - 1}\right) - 3\ln\left(\frac{\cos\frac{\psi}{2}}{\cos\frac{\psi}{2}}\cdot\frac{1 - \tan\frac{\psi}{2}}{1 + \tan\frac{\psi}{2}}\right) + 6\sec\psi\tan\psi\right] \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\left(3\ln\frac{\cot\frac{\psi}{2} + 1}{\cot\frac{\psi}{2} - 1} - 3\ln\frac{1 - \tan\frac{\psi}{2}}{1 + \tan\frac{\psi}{2}} + 6\sec\psi\tan\psi\right) \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\left[3\ln\left(\frac{\cot\frac{\psi}{2} + 1}{\cot\frac{\psi}{2} - 1}\cdot\frac{1 + \tan\frac{\psi}{2}}{1 - \tan\frac{\psi}{2}}\right) + 6\sec\psi\tan\psi\right] \\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\left(3\ln\frac{\cot\frac{\psi}{2} + \tan\frac{\psi}{2} + 2}{\cot\frac{\psi}{2} + \tan\frac{\psi}{2} - 2} + 6\sec\psi\tan\psi\right)
\end{aligned}$$

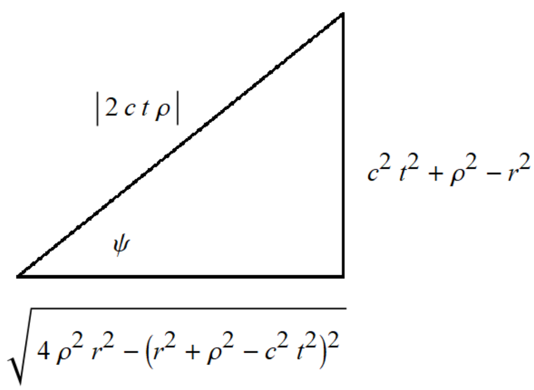
Note that

$$\cot\frac{\psi}{2} + \tan\frac{\psi}{2} = \frac{1}{\tan\frac{\psi}{2}} + \tan\frac{\psi}{2} = \frac{1 + \tan^2\frac{\psi}{2}}{\tan\frac{\psi}{2}} = \frac{\sec^2\frac{\psi}{2}}{\tan\frac{\psi}{2}} = \frac{\frac{1}{\cos^2\frac{\psi}{2}}}{\frac{\sin\frac{\psi}{2}}{\cos\frac{\psi}{2}}} = \frac{1}{\sin\frac{\psi}{2}\cos\frac{\psi}{2}} = \frac{1}{\frac{1}{2}\sin\psi} = \frac{2}{\sin\psi}.$$

Continue the simplification of I.

$$\begin{aligned}
I &= \frac{r^3\sin^2\beta\cos\beta}{6}\left(3\ln\frac{\frac{2}{\sin\psi}+2}{\frac{2}{\sin\psi}-2}+6\sec\psi\tan\psi\right)\\
&= \frac{r^3\sin^2\beta\cos\beta}{6}\left(3\ln\frac{1+\sin\psi}{1-\sin\psi}+6\sec\psi\tan\psi\right)\\
&= \frac{r^3\sin^2\beta\cos\beta}{6}(3\cdot 2\tanh^{-1}\sin\psi+6\sec\psi\tan\psi)\\
&= r^3\sin^2\beta\cos\beta(\tanh^{-1}\sin\psi+\sec\psi\tan\psi)
\end{aligned}$$

The definition of ψ implies the following right triangle.



From this triangle we have these three relationships.

$$\sin\psi = \frac{c^2t^2+\rho^2-r^2}{2ct\rho} \qquad \sec\psi = \frac{|2ct\rho|}{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}} \qquad \tan\psi = \frac{c^2t^2+\rho^2-r^2}{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}}$$

From the previous one we have these two relationships.

$$\cos\beta = \frac{r^2+\rho^2-c^2t^2}{2\rho r} \qquad \sin\beta = \frac{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}}{2\rho r}$$

Therefore,

$$\begin{aligned}
I &= r^3\left[\frac{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}{4\rho^2r^2}\right]\left(\frac{r^2+\rho^2-c^2t^2}{2\rho r}\right)\\
&\quad\times\left[\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}+\left(\frac{|2ct\rho|}{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}}\right)\left(\frac{c^2t^2+\rho^2-r^2}{\sqrt{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}}\right)\right]\\
&= \frac{(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]}{8\rho^3}\left[\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}+\frac{2c\rho|t|(c^2t^2+\rho^2-r^2)}{4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2}\right]\\
&= \frac{(r^2+\rho^2-c^2t^2)[4\rho^2r^2-(r^2+\rho^2-c^2t^2)^2]}{8\rho^3}\tanh^{-1}\frac{c^2t^2+\rho^2-r^2}{2ct\rho}+\frac{c|t|[\rho^4-(r^2-c^2t^2)^2]}{4\rho^2}.
\end{aligned}$$